CHAPTER 12

Regression with Time-Series Data: Nonstationary Variables

LEARNING OBJECTIVES

Based on the material in this chapter, you should be able to

1. Explain the differences between stationary and nonstationary time-series processes.
2. Describe the general behavior of an autoregressive process and a random walk process.
3. Explain why we need "unit root" tests, and state implications of the null and alternative hypotheses.
4. Explain what is meant by the statement that a series is "integrated of order one" or I(1).
5. Perform Dickey–Fuller and augmented Dickey–Fuller tests for stationarity.
6. Explain the meaning of a "spurious regression."
7. Explain the concept of cointegration and test whether two series are cointegrated.
8. Explain how to choose an appropriate model for regression analysis with time-series data.

KEYWORDS

autoregressive process
cointegration
Dickey–Fuller test
difference stationary
mean reversion
nonstationary
order of integration
random walks
random walk with drift
spurious regressions
stationary
stochastic process
stochastic trend
tau statistic
trend stationary
unit root tests

The analysis of time-series data is of vital interest to many groups, such as macroeconomists studying the behavior of national and international economies, finance economists analyzing the stock market, and agricultural economists predicting supplies and demands for agricultural products. For example, if we are interested in forecasting the growth of gross domestic product or inflation, we look at various indicators of economic performance and consider their behavior over recent years. Alternatively, if we are interested in a particular business, we analyze the history of the industry in an attempt to predict potential sales. In each of these cases, we are analyzing time-series data.

We worked with time-series data in Chapter 9 and discovered how regression models for these data often have special characteristics designed to capture their dynamic nature. We saw how including lagged values of the dependent variable or explanatory variables as regressors, or considering lags in the errors, can be used to model dynamic relationships. We showed how autoregressive distributed lag (ARDL) models can be used for forecasting and for computing dynamic multpliers. An important assumption that was maintained throughout Chapter 9 was that the variables are **stationary** and **weakly dependent**. They have means and variances that do not change over time, and autocorrelations that depend on the time between observations, not on the actual time of the observation. Also, their autocorrelations die out, eventually becoming negligible, as the distance between observations increases. There are, however, many economic time series that are not stationary—their means and/or variances change over time—and which exhibit strong dependence—their autocorrelations do not die out or they decline very slowly. In this chapter, we investigate the nature of **nonstationary** variables, examine the consequences of using them in regression analysis, introduce tests for stationarity, and learn how to model regression relationships that involve nonstationary variables. One important new concept that we encounter and which has a bearing on our choice of a regression model is **cointegration**. The widespread use of cointegration and its relevance for many economic time series led to a joint award of the 2003 Nobel Prize in Economics to its developer Clive W.J. Granger.[1]

12.1 Stationary and Nonstationary Variables

To illustrate the characteristics of nonstationary variables and appreciate their widespread relevance, we begin by examining some important economic variables for the U.S. economy.

EXAMPLE 12.1 | Plots of Some U.S. Economic Time Series

On the left-hand side of Figure 12.1, we display plots of real gross domestic product (a measure of aggregate economic production), the annual inflation rate (*INF*) (a measure of changes in the aggregate price level), the federal funds rate (*FFR*) (the interest rate on overnight loans between banks), and the three-year bond rate (*BR*) (interest rate on a financial asset to be held for three years). The data on gross domestic product (GDP) are quarterly from 1984Q1 to 2016Q4; they can be found in the data file *gdp5*. The data on inflation and the two interest rates are monthly from 1954M8 to 2016M12; they are stored in the data file *usdata5*. *FFR* and *BR* are used for several examples later in the Chapter. Observe how the GDP variable displays upward trending behavior, while the other series "wander up and down" with no discernable pattern or trend.

The figures on the right-hand side of Figure 12.1 are the changes of the corresponding variables on the left-hand side. Recall that we used changes in variables for several of our examples and exercises in Chapter 9. The change in a variable is a particularly important concept used repeatedly in this chapter; it is worth dwelling on its definition. The change in a variable y_t, also known as its **first difference**, is given by $\Delta y_t = y_t - y_{t-1}$. It is the change in the value of the variable y from period $t-1$ to period t. The time series of the changes on the right-hand side of Figure 12.1 display behavior that can be described as irregular ups and downs or more like fluctuations. Changes in the inflation rate and the two interest rates appear to fluctuate around a constant value, approximately zero. Changes in the GDP variable appear to fluctuate around a nonzero value, with a big dip at the time of the global financial crisis. The first question we address in this chapter is: Which data series represent stationary variables and which are observations on nonstationary variables?

[1] See https://www.britannica.com/biography/Clive-Granger. The corecipient of the 2003 Nobel Prize in Economics was Robert F. Engle whose contribution we consider in Chapter 14.

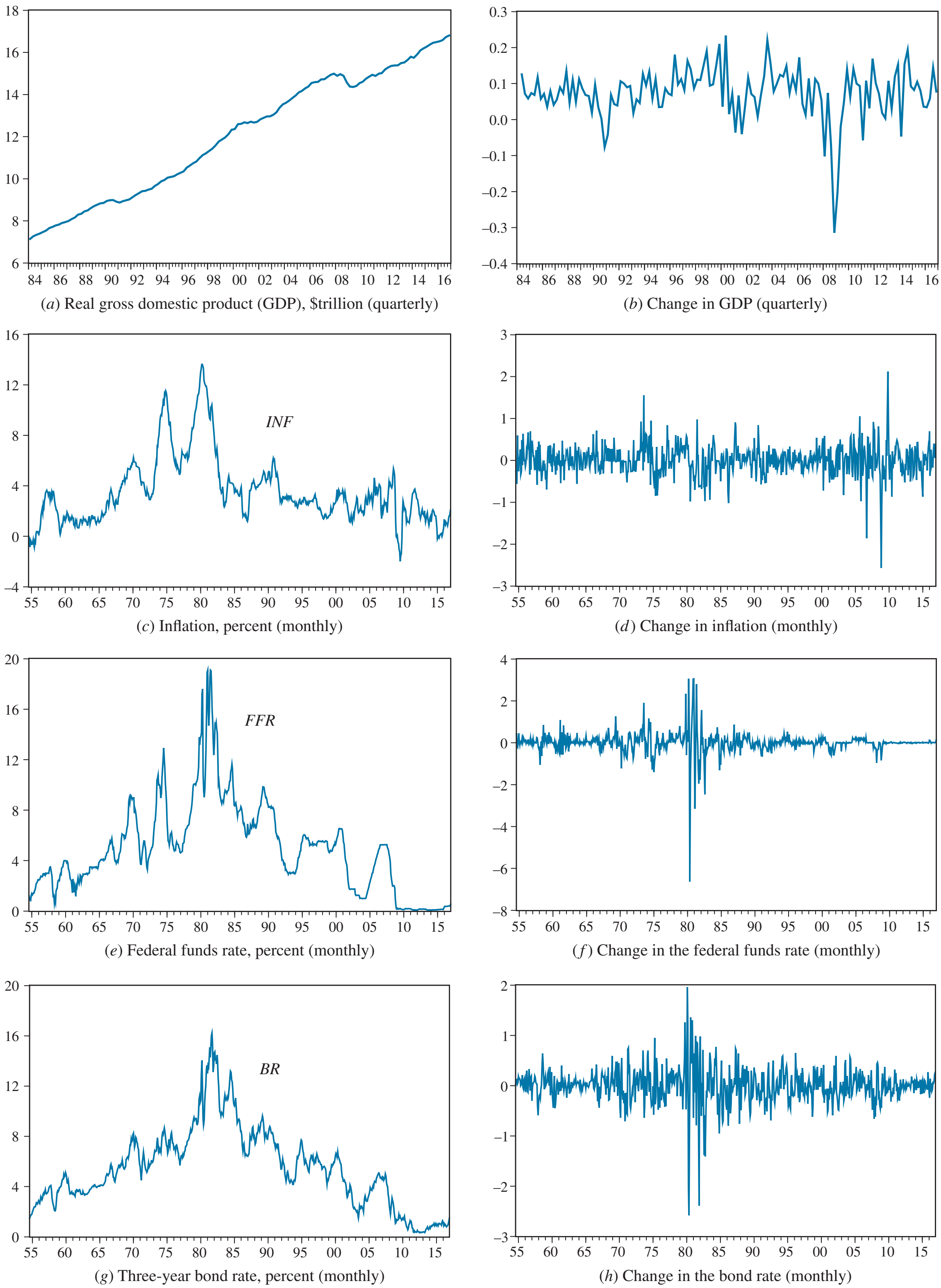

FIGURE 12.1 **U.S. Economic Time Series.**

Recall that a stationary time series y_t has mean and variance that are constant over time, and that the covariance (and autocorrelations) between two values from the series depends only on the length of time separating the two values, and not on the actual times at which the values are observed, that is,

$$E(y_t) = \mu \qquad \text{(constant mean)} \tag{12.1a}$$

$$\text{var}(y_t) = \sigma^2 \qquad \text{(constant variance)} \tag{12.1b}$$

$$\text{cov}(y_t, y_{t+s}) = \text{cov}(y_t, y_{t-s}) = \gamma_s \qquad \text{(covariance depends on } s\text{, not } t\text{)} \tag{12.1c}$$

Let us focus on the first condition, that of a constant mean. To investigate whether the means of the series in Figure 12.1 change over time, we divide the observations into two approximately equal subsamples, and compute the sample means for each of these subsamples. They are reported in Table 12.1. Examining the entries in this Table, as well as the plots in Figure 12.1, it is clear that the means of the variables expressed in terms of their original levels do change over time. In Figure 12.1(a), GDP exhibits a clear trend upward leading to a larger mean in the second half of the sample. The other three variables (Figures 12.1(c), (e), and (g)) wander up and then down, making the sample means very sensitive to the period selected. When the sample is divided into two equal parts, more large values appear in the first half of the sample, making the means in that half larger than those in the second half. These characteristics are typical of nonstationary variables. On the other hand, the first differences of the variables (their changes) in Figures 12.1(b), (d), (f), and (h) do not exhibit obvious trends. Their means for the two subsamples are similar in magnitude, particularly when viewed relative to magnitude of their quarter-to-quarter fluctuations. Having a constant mean and fluctuations in the series that tend to return to the mean are characteristics of stationary variables. They have the property of **mean reversion**.

Another characteristic of nonstationary variables is that their sample autocorrelations remain large at long lags. Stationary weakly dependent series have autocorrelations that cut off or tend to decline geometrically, dying out at long lags. The sample autocorrelations of nonstationary series exhibit **strong dependence**. They decline linearly rather than geometrically and are still significant at long lags. As an example, in Figure 12.2, the correlograms for GDP and its change are displayed. The autocorrelations for GDP decline very slowly and continue to be significant, well above the 5% significance bound of 0.17, even at lag 24, an indication that GDP is nonstationary. On the other hand, for the change in GDP, only the first two autocorrelations are significant before the remainder become negligible, suggesting that ΔGDP is stationary.

TABLE 12.1 **Sample Means of Time Series Shown in Figure 12.1**

		Sample Periods	
	GDP	1948Q2 to 2000Q3	2000Q4 to 2016Q4
Variable	*INF*, *BR*, *FFR*	1954M8 to 1985M10	1985M11 to 2016M12
Real GDP (a)		9.56	14.68
Inflation rate (c)		4.42	2.59
Federal funds rate (e)		6.20	3.65
Bond rate (g)		6.56	4.29
Change in GDP (b)		0.083	0.065
Change in the inflation rate (d)		0.01	−0.003
Change in the federal funds rate (f)		0.02	−0.02
Change in the bond rate (h)		0.02	−0.02

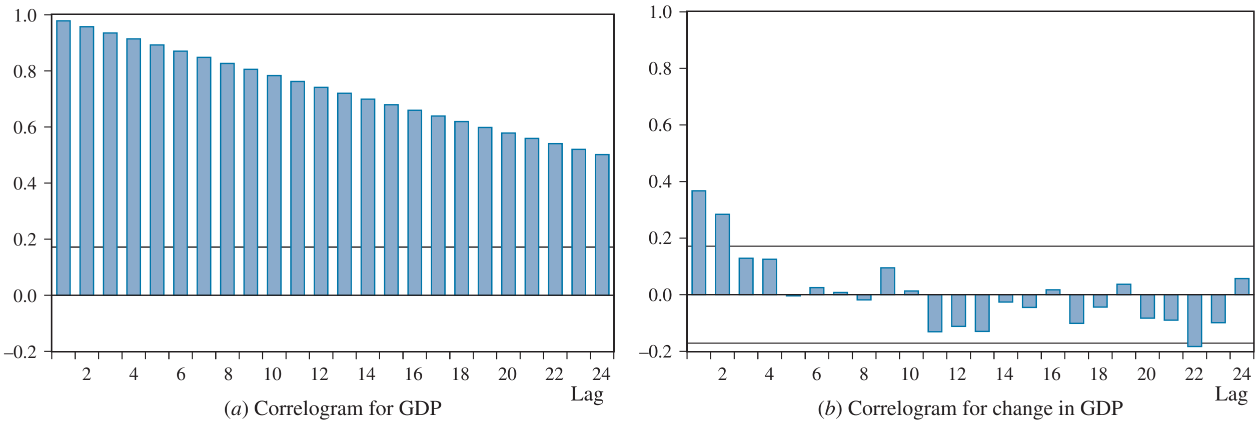

FIGURE 12.2 **Correlograms for GDP and the change in GDP.**

Plotting a series, examining whether its mean changes over time, and checking its sample autocorrelations give some indication of whether a series is stationary or nonstationary, but these checks are not conclusive, and they lack the rigor of a formal hypothesis test. Also, our discovery that series with nonstationary characteristics have stationary characteristics after first differencing is a common occurrence, but it is not universal and it needs verification. Formal testing for stationarity is introduced in Section 12.3. Before then, we discuss modeling series with trends and the consequences of nonstationarity for least-squares regressions.

12.1.1 Trend Stationary Variables

In Example 12.1, we saw how GDP has a definite trend, making it nonstationary, and that the other variables—inflation and the two interest rates—tend to wander up and down, another characteristic of nonstationary variables. Nonstationary variables that wander up and down, trending in one direction and then the other, are said to possess a **stochastic trend**. Definite trends, upward or downward, can be attributable to a stochastic trend or a **deterministic trend**, and sometimes both. Variables that are stationary after "subtracting out" a deterministic trend are called **trend stationary**. In this Section, we consider the notion of a deterministic trend, how it relates to the concept of trend stationarity, and the modeling of regression relationships involving trend stationary variables. Stochastic trends are introduced in Section 12.1.3.

The simplest model for a deterministic trend for a variable y is the linear trend model

$$y_t = c_1 + c_2 t + u_t \tag{12.2}$$

where $t = 1, 2, \dots, T$. If we focus just on the trend and assume any change in the error is zero $\left(\Delta u_t = u_t - u_{t-1} = 0\right)$, then the coefficient c_2 gives the change in y from a one period to the next

$$y_t - y_{t-1} = \left(c_1 + c_2 t\right) - \left[c_1 + c_2 (t-1)\right] + \Delta u_t = c_2$$

The "time variable" t does not necessarily have to start at "1" and increase in increments of "1". Redefining it using a linear transformation, say $t^* = a + bt$, simply changes the values for c_1 and c_2 and changes the interpretation of c_2 if $b \neq 0$. The trend $c_1 + c_2 t$ is called a deterministic trend because it does not contain a stochastic (random) component. The variable y_t is trend stationary if its fluctuations around this trend are stationary. Since these fluctuations are given by changes in the error term

$$u_t = y_t - \left(c_1 + c_2 t\right) \tag{12.3}$$

y_t is trend stationary if u_t is stationary.

When y_t is trend stationary, we can use least squares to find estimates $\hat{c}_1$ and $\hat{c}_2$ from (12.2) and then convert the trend stationary variable y_t to a stationary variable $\hat{u}_t$ by removing the trend:

$$\hat{u}_t = y_t - \left(\hat{c}_1 + \hat{c}_2 t\right) \tag{12.4}$$

If we are considering a regression or an ARDL model involving two trend stationary variables, say y_t and x_t, then, after their trends have been removed, making them stationary, their relationship can be estimated within the framework of Chapter 9.

To explore this notion further, suppose $y_t = c_1 + c_2 t + u_t$ and $x_t = d_1 + d_2 t + v_t$ are trend stationary variables; both u_t and v_t are stationary. To estimate a relationship between y_t and x_t, we first remove their trends: $\tilde{y}_t = y_t - \left(\hat{c}_1 + \hat{c}_2 t\right)$ and $\tilde{x}_t = x_t - \left(\hat{d}_1 + \hat{d}_2 t\right)$ where $\hat{c}_1, \hat{c}_2, \hat{d}_1$ and $\hat{d}_2$ are the least-squares estimates from the respective trends. We have used the notation $\tilde{y}_t$ and $\tilde{x}_t$ instead of $\hat{u}_t$ and $\hat{v}_t$ in line with that used in the FWL theorem introduced in Section 5.2.4. If we hypothesize that changes in y around its trend are related to changes in x around its trend, without any lags, a suitable linear model is

$$\tilde{y}_t = \beta\tilde{x}_t + e_t \tag{12.5}$$

An intercept can be omitted because $\tilde{y}_t$ and $\tilde{x}_t$ are OLS residuals with zero means. Now, we know from the FWL theorem that the OLS estimate of β from (12.5) is identical to the OLS estimate of β from the equation

$$y_t = \alpha_1 + \alpha_2 t + \beta x_t + e_t \tag{12.6}$$

Thus, when y and x are trend stationary, we can estimate a relationship between them by first removing the trends or by including a trend variable in the equation.

With trend stationary variables in more general ARDL models, we can proceed in a similar way, estimating either

$$\tilde{y}_t = \sum_{s=1}^{p}\theta_s\tilde{y}_{t-s} + \sum_{r=0}^{q}\delta_r\tilde{x}_{t-r} + e_t \tag{12.7}$$

or

$$y_t = \alpha_1 + \alpha_2 t + \sum_{s=1}^{p}\theta_s y_{t-s} + \sum_{r=0}^{q}\delta_r x_{t-r} + e_t \tag{12.8}$$

Assuming we create $\tilde{y}_{t-s}$ and $\tilde{x}_{t-r}$ by lagging $\tilde{y}_t$ and $\tilde{x}_t$, not by separately detrending every lag of y and x, there will be some slight differences in the estimates from (12.7) and (12.8).

Because trend stationary variables do not introduce any special problems providing a trend is included or the variables are detrended, they are often simply referred to as "stationary," although, strictly speaking, they are not stationary because their means change over time. Also, it is important not to ignore any trend. Estimating the model $y_t = \alpha_1 + \beta x_t + e_t$ when both y_t and x_t have deterministic trends can suggest a significant relationship between y_t and x_t even when none exists.

It is useful to pause at this point to emphasize what we have established and what we have not yet covered. We have discovered that regression relationships between trend stationary variables can be modeled by removing the deterministic trend from the variables, making them stationary, or by including the deterministic trend directly in the equation. What we have not yet covered is how to distinguish between deterministic trends and stochastic trends and how to model regression relationships between nonstationary variables with stochastic trends. In Example 12.1, GDP had an obvious trend. We do not yet know whether this trend is deterministic or stochastic, or how it should be modeled within a regression framework. We address these questions in the upcoming sections, but first it is useful to note that the linear trend in (12.2) is not the only possible deterministic trend, and to give an example.

Other Trends Another popular trend is one where, on average, a variable is growing at a constant *percentage* rate. If we momentarily ignore the error term, then, for a proportional change a_2, we have $y_t = y_{t-1} + a_2 y_{t-1}$, or, in percentage terms,

$$100 \times \left(\frac{y_t - y_{t-1}}{y_{t-1}}\right) = 100a_2$$

Recognizing that $(y_t - y_{t-1})/y_{t-1}$ can be approximated by $\Delta\ln(y_t) = \ln(y_t) - \ln(y_{t-1})$, we have

$$\ln(y_t) - \ln(y_{t-1}) \cong \%\Delta y_t = 100a_2$$

A model with this property, with an error term included, is

$$\ln(y_t) = a_1 + a_2 t + u_t \tag{12.9}$$

In this case. the deterministic trend for y_t is $\exp(a_1 + a_2 t)$, and $\ln(y_t)$ will be trend stationary if u_t is stationary. This model was introduced earlier in Section 4.5.1 in the context of modeling increases in wheat yield that are attributable to technological change. It may pay to go back and reread that Section now; it will give you more insights into the constant growth rate model.

The deterministic trend models in (12.2) and (12.9) are the most common, but others are possible. In Section 4.4.2, the cubic trend $y_t = \beta_1 + \beta_2 t^3 + e_t$ was used to model wheat yield. In Exercises 5.21 and 5.22, the interaction variable $TREND \times RAIN$ was included. A quadratic trend was used to model a decreasing and then increasing income share in Exercises 6.28 and 6.29. However, most deterministic trends tend to be continuously increasing or decreasing in which case quadratic or cubic trends that eventually turn up or down may not be well suited. A restricted range of the curve may fit the data well for the sample period, but outside this range a quadratic or cubic may be unrealistic. For this reason, the deterministic trends implied by (12.2) and (12.9) are the most popular.

EXAMPLE 12.2 | A Deterministic Trend for Wheat Yield

Scientists are continually working on ways to increase global food production to keep pace with a growing world population. One small contribution to this effort is the work of agronomists who develop new varieties of wheat to increase wheat yield. In the Toodyay Shire of Western Australia, we expect wheat yield to be trending upward over time reflecting the development of new varieties. However, wheat growing in Western Australia is a risky business. Its success depends heavily on rainfall, which is not always reliable. Thus, we expect yield to fluctuate around an increasing trend. Data on annual wheat yield and rainfall during the growing season for the Toodyay Shire, from 1950 to 1997, can be found in the data file *toodyay5*. For wheat yield, we use the constant growth rate trend $\ln(YIELD_t) = a_1 + a_2 t + u_t$. The observations for $\ln(YIELD_t)$ are plotted in Figure 12.3(a), along with the linear trend line. The observations fluctuate around the increasing trend with a particularly bad year in 1969. Examining the rainfall data in Figure 12.3(b), we

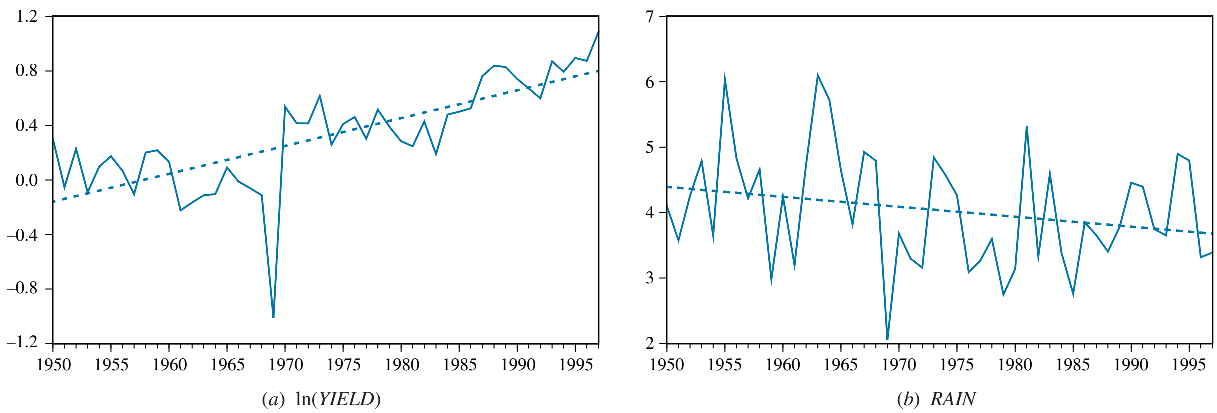

FIGURE 12.3 **Plots of time series for wheat yield example.**

discover there is a slight downward trend and very little rainfall in 1969.

It turns out that there are decreasing returns to rainfall and so we include $RAIN^2$ as well as $RAIN$ in the model, leading to the following estimated equation

$$\begin{aligned}\ln(YIELD_t) &= \underset{(se)}{-2.510} + \underset{(0.00252)}{0.01971t} + \underset{(0.290)}{1.149RAIN_t} \\ &\quad - \underset{(0.0346)}{0.1344RAIN_t^2} + \hat{e}_t \end{aligned} \tag{12.10}$$

The other alternative is to detrend ln($YIELD$), $RAIN$, and $RAIN^2$ and to estimate the detrended model. First, estimating the trends, we obtain

$$\widehat{\ln(YIELD_t)} = -0.1801 + \underset{(0.00276)}{0.02044\,t} \quad \text{(se)}$$

$$\widehat{RAIN_t} = 4.408 - \underset{(0.00891)}{0.01522\,t} \quad \text{(se)}$$

$$\widehat{RAIN_t^2} = 20.35 - \underset{(0.0747)}{0.1356\,t} \quad \text{(se)}$$

The first two equations describe the trend lines in Figure 12.3. After computing $RRAIN_t = RAIN_t - \widehat{RAIN_t}$, $RRAIN2_t = RAIN_t^2 - \widehat{RAIN_t^2}$, and $RLYIELD_t = \ln(YIELD_t) - \widehat{\ln(YIELD_t)}$, we obtain

$$\widehat{RLYIELD_t} = \underset{(0.284)}{1.149RRAIN_t} - \underset{(0.0339)}{0.1344RRAIN2_t} \quad \text{(se)} \tag{12.11}$$

Notice the estimates in (12.10) and (12.11) are identical, but the standard errors are not. The standard error discrepancy arises from the different degrees of freedom used to estimate the error variance. In (12.10), it is $48 - 4 = 44$; in (12.11), it is $48 - 2 = 46$. We can correct the standard errors in (12.11) by multiplying them by $\sqrt{46/44} = 1.022$. In large samples, the difference will be negligible. The legitimacy of the estimates in (12.10) and (12.11) depends on the assumption that ln($YIELD$), $RAIN$, and $RAIN^2$ are trend stationary. This assumption can be checked using the hypothesis testing machinery that is developed in Section 12.3 (see Exercise 12.16).

12.1.2 The First-Order Autoregressive Model

To develop a framework for modeling nonstationary variables that possess a stochastic trend, we begin by revising the first-order autoregressive AR(1) model that was introduced in Chapter 9.

The econometric model generating a time-series variable y_t is called a **stochastic** or **random process**. A sample of observed y_t values is called a particular **realization** of the **stochastic process**. It is one of many possible paths that the stochastic process could have taken. Univariate time-series models are examples of stochastic processes where a single variable y is related to past values of itself and current and past error terms. In contrast to regression modeling, univariate time-series models do not contain any explanatory variables (no x's).

The AR(1) model is a useful univariate time-series model for explaining the difference between stationary and nonstationary series. We first consider an AR(1) model with a zero mean given by

$$y_t = \rho y_{t-1} + v_t, \quad |\rho| < 1 \tag{12.12}$$

where the errors v_t are independent, with zero mean and constant variance σ_v^2, and may be normally distributed. In the context of time-series models, the errors are sometimes known as "shocks" or "innovations." As we will see, the assumption $|\rho| < 1$ implies that y_t is stationary. The AR(1) process shows that each realization of the random variable y_t contains a proportion ρ of last period's value y_{t-1} plus an error v_t drawn from a distribution with mean zero and variance σ_v^2. Since we are concerned with only one lag, the model is described as an autoregressive model of order one. In general, an AR(p) model includes lags of the variable y_t up to y_{t-p}. An example of an AR(1) time series with $\rho = 0.7$ and independent $N(0, 1)$ random errors is shown in Figure 12.4a. Note that the data have been artificially generated. Observe how the time series fluctuates around zero and has no trend-like behavior, a characteristic of stationary series.

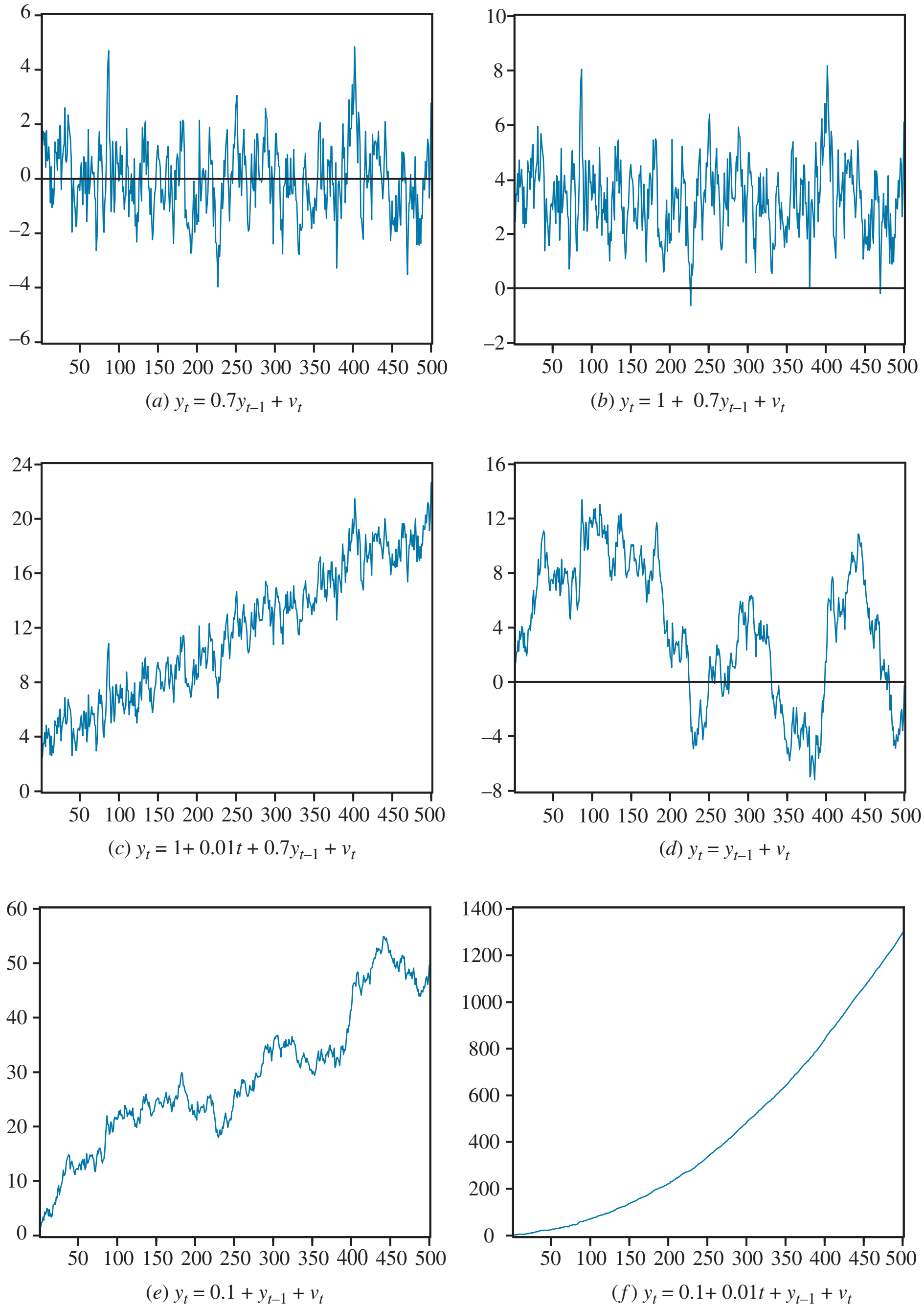

FIGURE 12.4 **Time-series models.**

The value "zero" is the constant mean of the series, and it can be determined by doing some algebra known as **recursive substitution**.[2] Consider the value of y at time $t = 1$, then its value at time $t = 2$, and so on. These values are

[2]An alternative to recursive substitution when the variable is stationary is to use the lag operator algebra discussed in Section 9.5.4.

$$y_1 = \rho y_1 + v_1$$
$$y_2 = \rho y_1 + v_2 = \rho(\rho y_0 + v_1) + v_2 = \rho^2 y_0 + \rho v_1 + v_2$$
$$\vdots$$
$$y_t = v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \cdots + \rho^t y_0$$

The mean of y_t is

$$E(y_t) = E(v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \cdots) = 0$$

since the error v_t has zero mean, and the value of $\rho^t y_0$ is negligible for a large t. In Appendix 9B, the variance was shown to be a constant $\sigma_v^2/(1-\rho^2)$, while the covariance between two errors s periods apart γ_s is $\sigma_v^2 \rho^s/(1-\rho^2)$. Thus, the AR(1) model in (12.12) is a classic example of a stationary process with a zero mean.

Real-world data rarely have a zero mean. We can introduce a nonzero mean μ by replacing y_t in (12.12) with $(y_t - \mu)$ as follows:

$$(y_t - \mu) = \rho(y_{t-1} - \mu) + v_t$$

which can then be rearranged as

$$y_t = \alpha + \rho y_{t-1} + v_t, \quad |\rho| < 1 \tag{12.13}$$

where $\alpha = \mu(1-\rho)$, that is, we can accommodate a nonzero mean in y_t by either working with the "demeaned" variable $(y_t - \mu)$ or introducing the intercept term α in the **autoregressive process** of y_t as in (12.13). Corresponding to these two ways, we describe the "de-meaned" variable $(y_t - \mu)$ as being stationary around zero, or the variable y_t as stationary around its mean value $\mu = \alpha/(1-\rho)$.

An example of a time series that follows this model, with $\alpha = 1$, $\rho = 0.7$ is shown in Figure 12.4(b). We have used the same values of the error v_t as in Figure 12.4(a), so the figure shows the added influence of the constant term. Note that the series now fluctuates around a nonzero value. This nonzero value is the constant mean of the series

$$E(y_t) = \mu = \alpha/(1-\rho) = 1/(1-0.7) = 3.33$$

Another extension to (12.12) is to consider an AR(1) model fluctuating around a linear trend $(\mu + \delta t)$. In this case, we let the "detrended" series $(y_t - \mu - \delta t)$ behave like an autoregressive model

$$(y_t - \mu - \delta t) = \rho[y_{t-1} - \mu - \delta(t-1)] + v_t, \quad |\rho| < 1$$

which can be rearranged as

$$y_t = \alpha + \rho y_{t-1} + \lambda t + v_t \tag{12.14}$$

where $\alpha = [\mu(1-\rho) + \rho\delta]$ and $\lambda = \delta(1-\rho)$. For $|\rho| < 1$, equation (12.14) is an example of a trend-stationary process. Figure 12.4(c) displays a plot of this process for parameters $\rho = 0.7$, $\alpha = 1$, and $\lambda = 0.01$. The detrended series $(y_t - \mu - \delta t)$ has a constant variance, and covariances that depend only on the time separating observations, not the time at which they are observed. In other words, the detrended series is stationary; y_t is stationary around the deterministic trend line $\mu + \delta t$.

12.1.3 Random Walk Models

Consider the special case of $\rho = 1$ in (12.12):

$$y_t = y_{t-1} + v_t \tag{12.15}$$

This model is known as the random walk model. Equation (12.15) shows that each realization of the random variable y_t contains last period's value y_{t-1} plus an error v_t. An example of a time series that can be described by this model is shown in Figure 12.4(d). These time series are called **random walks** because they appear to wander slowly upward or downward with no real pattern; the values of sample means calculated from subsamples of observations will be dependent on the sample period, a characteristic of nonstationary series.

We can understand the "wandering" behavior of random walk models by doing some recursive substitution.

$$\begin{aligned} y_1 &= y_0 + v_1 \\ y_2 &= y_1 + v_2 = (y_0 + v_1) + v_2 = y_0 + \sum_{s=1}^{2} v_s \\ &\vdots \\ y_t &= y_{t-1} + v_t = y_0 + \sum_{s=1}^{t} v_s \end{aligned}$$

The random walk model contains an initial value y_0 (often set to zero because it is so far in the past that its contribution to y_t is negligible) plus a component that is the sum of the past stochastic terms $\sum_{s=1}^{t} v_s$. This latter component is called the **stochastic trend**. This term arises because a stochastic component v_t is added for each time t, and because it causes the time series to trend in unpredictable directions. If the variable y_t is subjected to a sequence of positive shocks, $v_t > 0$, followed by a sequence of negative shocks, $v_t < 0$, it will have the appearance of wandering upward, then downward.

We have used the fact that y_t is a sum of errors to explain graphically the nonstationary nature of the random walk. We can also use it to show algebraically that the conditions for stationarity do not hold. Recognizing that the v_t are independent with zero means and identical variances σ_v^2, taking the expectation and the variance of y_t yields, for a fixed initial value y_0,

$$\begin{aligned} E(y_t) &= y_0 + E(v_1 + v_2 + \cdots + v_t) = y_0 \\ \text{var}(y_t) &= \text{var}(v_1 + v_2 + \cdots + v_t) = t\sigma_v^2 \end{aligned}$$

The random walk has a mean equal to its initial value and a variance that increases over time, eventually becoming infinite. Although the mean is constant, the increasing variance implies that the series may not return to its mean, and so sample means taken for different periods are not the same.

Another nonstationary model is obtained by adding a constant term to (12.15):

$$y_t = \delta + y_{t-1} + v_t \tag{12.16}$$

This model is known as the **random walk with drift**. Equation (12.16) shows that each realization of the random variable y_t contains an intercept (the drift component δ) plus last period's value y_{t-1} plus the error v_t. An example of a time series that can be described by this model (with $\delta = 0.1$) is shown in Figure 12.4(e). Notice how the time-series data appear to be "wandering" as well as "trending" upward. In general, random walk with drift models show definite trends either upward (when the drift δ is positive) or downward (when the drift δ is negative).

Again, we can get a better understanding of this behavior by applying recursive substitution:

$$\begin{aligned} y_1 &= \delta + y_0 + v_1 \\ y_2 &= \delta + y_1 + v_2 = \delta + (\delta + y_0 + v_1) + v_2 = 2\delta + y_0 + \sum_{s=1}^{2} v_s \\ &\vdots \\ y_t &= \delta + y_{t-1} + v_t = t\delta + y_0 + \sum_{s=1}^{t} v_s \end{aligned}$$

The value of y at time t is made up of an initial value y_0, the stochastic trend component $\left(\sum_{s=1}^{t} v_s\right)$, and now a deterministic trend component $t\delta$. It is called a deterministic trend because a fixed value

δ is added for each time t. The variable y wanders up and down as well as increases by a fixed amount at each time t. The mean and variance of y_t are

$$E(y_t) = t\delta + y_0 + E(v_1 + v_2 + v_3 + \cdots + v_t) = t\delta + y_0$$
$$\text{var}(y_t) = \text{var}(v_1 + v_2 + v_3 + \cdots + v_t) = t\sigma_v^2$$

In this case, both the constant mean and constant variance conditions for stationarity are violated.

We can extend the random walk model even further by adding a time trend:

$$y_t = \alpha + \delta t + y_{t-1} + v_t \tag{12.17}$$

An example of a time series that can be described by this model (with $\alpha = 0.1; \delta = 0.01$) is shown in Figure 12.4(f). Note how the addition of a time-trend variable t strengthens the trend behavior. We can see the amplification using the same algebraic manipulation as before:

$$y_1 = \alpha + \delta + y_0 + v_1$$
$$y_2 = \alpha + \delta 2 + y_1 + v_2 = \alpha + 2\delta + (\alpha + \delta + y_0 + v_1) + v_2 = 2\alpha + 3\delta + y_0 + \sum_{s=1}^{2} v_s$$
$$\vdots$$
$$y_t = \alpha + \delta t + y_{t-1} + v_t = t\alpha + \left(\frac{t(t+1)}{2}\right)\delta + y_0 + \sum_{s=1}^{t} v_s$$

where we have used the formula for a sum of an arithmetic progression,

$$1 + 2 + 3 + \cdots + t = t(t+1)/2$$

The additional term has the effect of strengthening the trend behavior.

To recap, we have considered the autoregressive class of models and have shown that they display properties of stationarity when $|\rho| < 1$. We have also discussed the random walk class of models when $\rho = 1$. We showed that random walk models display properties of nonstationarity. Now, go back and compare the real-world data in Figure 12.1 with those in Figure 12.4. Ask yourself what models might have generated the different data series in Figure 12.1. In the next few sections we shall consider how to test which series in Figure 12.1 exhibit properties associated with stationarity, as well as which series exhibit properties associated with nonstationarity.

12.2 Consequences of Stochastic Trends

In Section 12.1.2, we noted that regressions involving variables with a deterministic trend, *and no stochastic trend*, did not present any difficulties providing the trend was included in the regression relationship, or the variables were detrended. Allowing for the trend was important because excluding it could lead to omitted variable bias. Now we consider the implications of estimating regressions involving variables with stochastic trends. In this context, because stochastic trends are the most prevalent source of nonstationarity, and they introduce special problems, when we refer to nonstationary variables, we will generally mean variables that are neither stationary nor trend stationary.

A consequence of proceeding with the regression involving nonstationary variables with stochastic trends is that OLS estimates no longer have approximate normal distributions in large samples. That means interval estimates and hypothesis tests will no longer be valid. Precision of estimation may not be what it seems to be and conclusions about relationships between variables could be wrong. One particular hazard is that two totally independent random walks can appear to have a strong linear relationship when none exists. Outcomes of this nature have been given the name **spurious regressions**.

EXAMPLE 12.3 | A Regression with Two Random Walks

To illustrate the spurious regression problem, consider the following two independent random walks:

$$rw_1 : y_t = y_{t-1} + v_{1t}$$
$$rw_2 : x_t = x_{t-1} + v_{2t}$$

where v_{1t} and v_{2t} are independent $N(0, 1)$ random errors. Two such series are shown in Figure 12.5(a)—the data are in the data file *spurious*. These series were generated independently and, in truth, have no relation to one another, yet when we plot them, as we have done in Figure 12.5(b), we see a positive relationship between them. If we estimate a simple regression of series one (rw_1) on series two (rw_2), we obtain the following results:

$$\begin{array}{ll} rw_{1t} = 17.818 + 0.842 rw_{2t}, & R^2 = 0.70 \\ (t) \qquad\qquad (40.837) & \end{array}$$

This result suggests that the simple regression model fits the data well ($R^2 = 0.70$) and that the estimated slope is significantly different from zero. In fact, the *t*-statistic is huge! These results are, however, completely meaningless, or spurious. The apparent significance of the relationship is false. It results from the fact that we have related one series with a stochastic trend to another series with another stochastic trend. In fact, these series have nothing in common, nor are they causally related in any way. Similar and more dramatic results are obtained when random walk with drift series are used in regressions. Typically the residuals from such regressions will be highly correlated. For this example, the LM test value to test for first-order autocorrelation (*p*-value in parenthesis) is 682.958 (0.000); a sure sign that there is a problem with the regression.

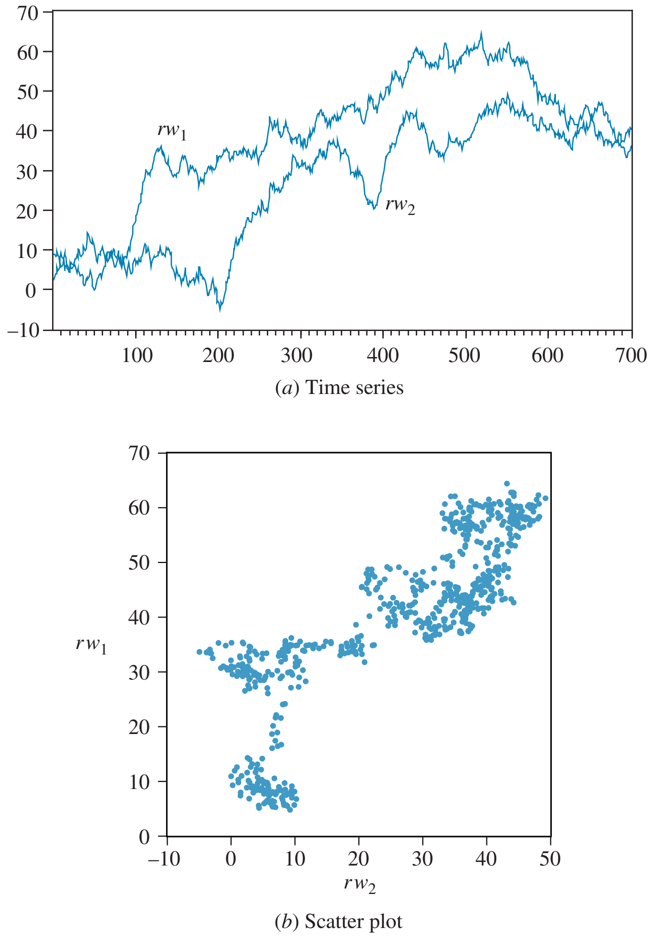

FIGURE 12.5 **Time series and scatter plot of two random walk variables.**

To summarize, when nonstationary time series are used in a regression model, the results may spuriously indicate a significant relationship when there is none. In these cases the least-squares estimator and least-squares predictor do not have their usual properties, and t-statistics are not reliable. Since many macroeconomic time series are nonstationary, it is particularly important to take care when estimating regressions with macroeconomic variables.

There are also important policy considerations for distinguishing between stationary and nonstationary variables. With nonstationary variables each error or shock v_t has a lasting effect, and these shocks accumulate. With stationary variables the effect of a shock eventually dies out and the variable returns to its mean. Whether a change in a macroeconomic variable has a permanent or transitory effect is essential information for policy makers.

How then can we test whether a series is stationary or nonstationary, and how do we conduct regression analysis with nonstationary data? The former is discussed in Section 12.3, while the latter is considered in Section 12.4.

12.3 Unit Root Tests for Stationarity

There are many tests for assessing whether a series is stationary or nonstationary. The most popular one, and the one that we discuss in detail, is the **Dickey–Fuller test** for a **unit root**. What do we mean by a "unit root"? Because you will hear this term frequently when nonstationary time series are being discussed, it is useful to digress for a moment to explain its origin.

12.3.1 Unit Roots

We have seen that in the AR(1) model $y_t = \alpha + \rho y_{t-1} + v_t$, y_t is stationary if $|\rho| < 1$ and nonstationary if $\rho = 1$. We also say that y_t has a unit root if $\rho = 1$, but to appreciate the origin of the term, we need to consider the more general AR(p) model $y_t = \alpha + \theta_1 y_{t-1} + \theta_2 y_{t-2} + \cdots + \theta_p y_{t-p} + v_t$. In this model, y_t is stationary if the roots of the polynomial equation

$$\varphi(z) = 1 - \theta_1 z - \theta_2 z^2 - \cdots - \theta_p z^p \tag{12.18}$$

are greater than one in absolute value. The roots are the values of z that satisfy the equation $\varphi(z) = 0$. When $p = 1$ and $y_t = \alpha + \theta_1 y_{t-1} + v_t$, we have $\varphi(z) = 1 - \theta_1 z = 0$, and $z = 1/\theta_1$. The condition for stationarity is $|z| > 1$, which is the same as $|\theta_1| < 1$. If, in (12.18), one of the roots is equal to one, then y_t is said to have a unit root. It has a stochastic trend and is nonstationary. When $p = 1$ and $\varphi(z) = 1 - \theta_1 z = 0$, then $z = 1$ implies $\theta_1 = 1$. Note that we have used θ_1 and ρ interchangeably for the AR(1) model. It is convenient to use θ_1 when considering the AR(1) process as a special case of an AR(p) process. Using ρ emphasizes that the coefficient of y_{t-1} in an AR(1) process is the first-order autocorrelation.

To summarize, if y_t has a unit root, it is nonstationary. For y_t to be stationary, the roots of (12.18) must be greater than one in absolute value. In the AR(1) model $y_t = \alpha + \rho y_{t-1} + v_t$, these conditions translate into $\rho = 1$ for the unit root and $|\rho| < 1$ for stationarity. In higher-order AR models, the conditions for a unit root and for stationarity, written in terms of the parameters $\theta_1, \theta_2, \ldots, \theta_p$, are more complicated. We explore these conditions for the AR(2) model in Exercise 12.1.

You might be wondering what happens if one of the roots of $\varphi(z)$ is less than one in absolute value. Or, in particular, what happens if $\rho > 1$ in the AR(1) process. In this case, y_t is nonstationary and explosive. Empirically, we do not observe time series that explode and so we restrict ourselves to unit roots and roots that imply a stationary process. In the Dickey–Fuller tests that follow the null hypothesis is that y_t has a unit root and the alternative is that y_t is stationary.

12.3.2 Dickey–Fuller Tests

There are three variations of the Dickey–Fuller test, each one designed for a different alternative hypothesis.

1. The alternative hypothesis is that y_t is stationary around a nonzero mean. An example of such a series is that depicted in Figure 12.4(b). In this case, the test equation includes an intercept but no trend term.
2. The alternative hypothesis is that y_t is stationary around a linear deterministic trend, like that depicted in Figure 12.4(d). Here, the test equation includes both intercept and trend terms.
3. The alternative hypothesis is that y_t is stationary around a zero mean as illustrated in Figure 12.4(a). Both intercept and trend are excluded from the test equation in this case.

The choice between these tests can be guided by the nature of the data, revealed by plotting the series against time. If it is not obvious from a plot which test is the most relevant—and it will not always be obvious—more than one test equation can be used to check the robustness of a test conclusion.

12.3.3 Dickey–Fuller Test with Intercept and No Trend

Consider a time series y_t that has no definite continuous trend upward or downward, and that is not obviously centered around zero. Suppose we wish to test whether this series is better represented by a stationary AR(1) process like that in Figure 12.4(b) or a nonstationary random walk like that in Figure 12.4(d). The nonstationary random walk is set up as the null hypothesis

$$H_0 : y_t = y_{t-1} + v_t \tag{12.19}$$

and the stationary AR(1) process becomes the alternative hypothesis

$$H_1 : y_t = \alpha + \rho y_{t-1} + v_t \quad |\rho| < 1 \tag{12.20}$$

Throughout, we assume the v_t are independent random errors with mean zero and variance σ_v^2, and that they are uncorrelated with the past values $y_{t-1}, y_{t-2}, \ldots$. Under H_1, the series fluctuates around a constant mean. Under H_0, it wanders upward and downward but does not exhibit a clear trend in either direction and does not tend to return to a constant mean.

An obvious way to specify the null hypothesis in terms of the parameters in the unrestricted alternative is $H_0 : \alpha = 0,\ \rho = 1$. A test for this purpose has been developed,[3] but it has become more common to simply specify the null as $H_0 : \rho = 1$. One way to justify omission of $\alpha = 0$ from H_0 is to recall that $\alpha = \mu(1 - \rho)$. If $\rho = 1$, then $\alpha = 0$, and so one can argue that testing $H_0 : \rho = 1$ is sufficient. Thus, we test for nonstationary in the AR(1) model $y_t = \alpha + \rho y_{t-1} + v_t$, by testing $H_0 : \rho = 1$ against the alternative $H_1 : |\rho| < 1$, or simply $H_1 : \rho < 1$. This one-sided (left tail) test is put into a more convenient form by subtracting y_{t-1} from both sides of (12.20) to obtain:

$$\begin{aligned} y_t - y_{t-1} &= \alpha + \rho y_{t-1} - y_{t-1} + v_t \\ \Delta y_t &= \alpha + (\rho - 1) y_{t-1} + v_t \\ &= \alpha + \gamma y_{t-1} + v_t \end{aligned} \tag{12.21}$$

where $\gamma = \rho - 1$ and $\Delta y_t = y_t - y_{t-1}$. Then, the hypotheses can be written either in terms of ρ or in terms of γ:

$$\begin{aligned} H_0 : \rho = 1 \quad &\Longleftrightarrow \quad H_0 : \gamma = 0 \\ H_1 : \rho < 1 \quad &\Longleftrightarrow \quad H_1 : \gamma < 0 \end{aligned} \tag{12.22}$$

[3] An advanced reference is Hamilton, J.D. (1994), *Time Series Analysis*, Princeton, p. 494.

TABLE 12.2 Critical Values for the Dickey–Fuller Test

Model	1%	5%	10%
$\Delta y_t = \gamma y_{t-1} + v_t$	−2.56	−1.94	−1.62
$\Delta y_t = \alpha + \gamma y_{t-1} + v_t$	−3.43	−2.86	−2.57
$\Delta y_t = \alpha + \lambda t + \gamma y_{t-1} + v_t$	−3.96	−3.41	−3.13
Standard normal critical values	−2.33	−1.65	−1.28

Note: These critical values are taken from R. Davidson and J. G. MacKinnon, *Estimation and Inference in Econometrics*, New York: Oxford University Press, 1993, p. 708.

Rejection of the null hypothesis that $\gamma = 0$ implies the series is stationary. A failure to reject H_0 suggests the series could be nonstationary, and we must be careful not to proceed to estimate a spurious regression.

To test the hypothesis in (12.22), we estimate the test equation (12.21) by OLS and examine the t-statistic for the hypothesis that $\gamma = 0$. Unfortunately, this t-statistic no longer has the t-distribution that we have used previously to test zero null hypotheses for regression coefficients. The problem arises because, when the null hypothesis is true, y_t is nonstationary and has a variance that increases as the sample size increases. This increasing variance alters the distribution of the usual t-statistic when H_0 is true. To recognize this fact, the statistic is often called a **τ (tau) statistic**, and its value must be compared to specially generated critical values. The critical values are different for each of the variations of the test described in Section 12.3.2. They are tabulated in Table 12.2.[4] Those for test equation (12.21) are given in the middle row. We reject $H_0 : \gamma = 0$ if $\tau \le \tau_c$, where $\tau = \hat{\gamma}/\text{se}(\hat{\gamma})$ is the OLS "t"-value for $H_0 : \gamma = 0$, and τ_c is a critical value from Table 12.2. In other words, we conclude y_t is stationary if τ is a sufficiently large negative number. Note that the Dickey–Fuller critical values are more negative than the standard normal critical values (shown in the last row). Thus, the τ-statistic must take larger (negative) values than usual for the null hypothesis of nonstationarity $(\gamma = 0)$ to be rejected in favor of the alternative of stationarity $(\gamma < 0)$.

There are many stationary series that are not adequately modeled by an AR(1) process. A natural question is how do we test for a unit root in a higher-order AR process. It can be shown[5] that testing for a unit root in the AR(p) process

$$y_t = \alpha + \theta_1 y_{t-1} + \theta_2 y_{t-2} + \cdots + \theta_p y_{t-p} + v_t$$

against the alternative that y_t is stationary, is equivalent to testing $H_0 : \gamma = 0$ against the alternative $H_1 : \gamma < 0$ in the model

$$\Delta y_t = \alpha + \gamma y_{t-1} + \sum_{s=1}^{p-1} a_s \Delta y_{t-s} + v_t \tag{12.23}$$

The original test equation is augmented by the lagged first differences $\Delta y_{t-1} = (y_{t-1} - y_{t-2})$, $\Delta y_{t-2} = (y_{t-2} - y_{t-3})$, … , $\Delta y_{t-p+1} = (y_{t-p+1} - y_{t-p})$. The test procedure for this case uses (12.23) as the test equation but otherwise proceeds just as before, rejecting $H_0 : \gamma = 0$ when $\tau = \hat{\gamma}/\text{se}(\hat{\gamma}) \le \tau_c$. The critical values are the same as those in Table 12.2. The test is referred to as the **augmented Dickey–Fuller test**. The choice for p can be based on similar criteria to

[4]Originally these critical values were tabulated by the statisticians David Dickey and Wayne Fuller. The values have since been refined, but in deference to the seminal work, unit root tests using these critical values have become known as Dickey–Fuller tests.

[5]See Exercise 12.1.

those described in Chapter 9 for choosing the order of an AR process. Sufficient lags should be included to eliminate autocorrelation in the errors. We can also use significance of the estimates of the a_s, which have their usual large-sample normal distributions, and the AIC and SC variable selection criteria. In practice, we always use the augmented Dickey–Fuller test (rather than the nonaugmented version) to ensure the errors are uncorrelated.

EXAMPLE 12.4 | Checking the Two Interest Rate Series for Stationarity

As an example, consider the two interest rate series—the federal funds rate FFR_t and the three-year bond rate BR_t—plotted in Figures 12.1(e) and (g), respectively. Both series exhibit wandering behavior, wandering up and then down with no discernible trend in either direction. We therefore suspect that they may be nonstationary variables. Using OLS to estimate the test equation (12.23) for each of these variables yields

$$\begin{aligned} \widehat{\Delta FFR_t} &= \underset{}{0.0580} - \underset{(-2.47)}{0.0118FFR_{t-1}} + \underset{(12.30)}{0.444\Delta FFR_{t-1}} \quad (\tau \text{ and } t) \\ &\quad \underset{(-4.05)}{-0.147\Delta FFR_{t-2}} \end{aligned}$$

$$\begin{aligned} \widehat{\Delta BR_t} &= 0.0343 - \underset{(-1.70)}{0.00635BR_{t-1}} + \underset{(11.95)}{0.426\Delta BR_{t-1}} \quad (\tau \text{ and } t) \\ &\quad \underset{(-6.43)}{-\,0.230\Delta BR_{t-2}} \end{aligned}$$

Two augmentation terms have been included for both variables. For FFR the number of augmentation terms that minimized the SC was 13—a very large number. However, checking the correlogram of the residuals, we find that including two lags of ΔFFR was sufficient to eliminate any major autocorrelation in the errors. For BR, two augmentation terms minimized the SC and were sufficient to eliminate any substantial error autocorrelation. The usual t or normal distributions can be used to assess the significance of the coefficients of the augmentation terms. Their large t-values confirm the decision to include two lags.

However, for checking stationarity, the usual t critical values and p-values cannot be used. Instead, we compare the two τ-values, $\tau = -2.47$ and $\tau = -1.70$ for the coefficients of FFR_{t-1} and BR_{t-1}, respectively, with a critical value from Table 12.2. For a 5% significance level, the relevant critical value is $\tau_{0.05} = -2.86$. The test for stationarity is a one-tail test with the null hypothesis of nonstationarity being rejected if $\tau \le -2.86$. Since $-2.47 > -2.86$ and $-1.70 > -2.86$, in both cases, we fail to reject H_0. There is insufficient evidence to suggest that FFR and BR are stationary.

12.3.4 Dickey–Fuller Test with Intercept and Trend

In Sections 12.1.2 and 12.1.3, we introduced two models where a time series y_t has a trend upward or downward. In one, illustrated in Figure 12.4(c), y_t was stationary around a linear trend and described by the process

$$y_t = \alpha + \rho y_{t-1} + \lambda t + v_t \qquad |\rho| < 1 \tag{12.24}$$

A time series that can be described by (12.24) is called trend stationary. The other model was a random walk with drift, illustrated in Figure 12.4(e):

$$y_t = \alpha + y_{t-1} + v_t \tag{12.25}$$

In this case y_t is nonstationary. The Dickey–Fuller test with intercept and trend is designed to discriminate between these two models. Equation (12.25) becomes the null hypothesis (H_0), and equation (12.24) is the alternative hypothesis (H_1). If the null hypothesis is rejected, we conclude y_t is trend stationary. Failure to reject H_0 suggests y_t is nonstationary, or at least there is insufficient evidence to prove otherwise.

Comparing (12.24) and (12.25) suggests a relevant null hypothesis is $H_0: \rho = 1, \lambda = 0$. However, like in Section 12.3.3, it has become more common to simply test $H_0: \rho = 1$ against the alternative $H_1: \rho < 1$. A rationale for doing so can be found by going back and checking equation (12.14). There we noted an alternative way of writing (12.24) is

$$\left(y_t - \mu - \delta t\right) = \rho\left(y_{t-1} - \mu - \delta(t-1)\right) + v_t, \qquad |\rho| < 1$$

where $\mu + \delta t$ is the deterministic trend, $\alpha = \mu(1-\rho) + \rho\delta$ and $\lambda = \delta(1-\rho)$. With these definitions of α and λ, setting $\rho = 1$ implies $\alpha = \delta$ and $\lambda = 0$, giving the random walk with drift in (12.25). As before, the test equation is obtained by subtracting y_{t-1} from both sides of (12.24) and adding augmentation terms to obtain

$$\Delta y_t = \alpha + \gamma y_{t-1} + \lambda t + \sum_{s=1}^{p-1} a_s \Delta y_{t-s} + v_t \tag{12.26}$$

We use the left-tail test $H_0: \gamma = 0$ versus $H_1: \gamma < 0$, rejecting H_0 when $\tau = \hat{\gamma}/\text{se}(\hat{\gamma})$ is less than or equal to a critical value selected from the third row of Table 12.2.

EXAMPLE 12.5 | Is GDP Trend Stationary?

From Figure 12.1(a), we noted that GDP shows a definite upward trend. We now ask whether it can be modeled as stationary around a linear deterministic trend, or whether it contains a stochastic trend component. Using these data to estimate (12.26) yields[6]

$$\begin{aligned} \widehat{\Delta GDP_t} &= 0.269 + 0.00249t - 0.0330\,GDP_{t-1} \\ (\tau \text{ and } t) &\qquad\qquad\qquad\quad (-2.00) \\ &\quad + 0.312\Delta GDP_{t-1} + 0.202\Delta GDP_{t-2} \\ &\qquad (3.58) \qquad\qquad (2.28) \end{aligned}$$

Two augmentation terms minimized the SC, eliminated major autocorrelation in the residuals, and had coefficient estimates significant at a 5% level. For assessing stationarity, we find $\tau = -2.00$, which is greater than the 5% critical value $\tau_{0.05} = -3.41$. Thus, we cannot reject the null hypothesis that GDP follows a nonstationary random walk with drift. There is insufficient evidence to conclude that GDP is trend stationary.

EXAMPLE 12.6 | Is Wheat Yield Trend Stationary?

In Example 12.2, we model wheat yield in the Toodyay Shire of Western Australia with a deterministic trend. To see whether this choice was justified we estimate the test equation

$$\begin{aligned} \widehat{\Delta \ln(YIELD_t)} &= -0.158 + 0.0167\,t - 0.745\ln(YIELD_{t-1}) \\ (\tau) &\qquad\qquad\qquad\qquad\quad (-5.24) \end{aligned}$$

In this case, no augmentation terms were necessary. The value $\tau = -5.24$ is less than the 5% critical value $\tau_{0.05} = -3.41$ and so, at this level of significance, we reject a null hypothesis of nonstationarity and conclude that $\ln(YIELD)$ is trend stationary.

12.3.5 Dickey–Fuller Test with No Intercept and No Trend

In its simplest form with no augmentation terms, this test is designed to test the null hypothesis of a random walk $H_0: y_t = y_{t-1} + v_t$ against the stationary AR(1) alternative $H_1: y_t = \rho y_{t-1} + v_t$, $|\rho| < 1$. Since y_t has a zero mean when H_1 is true, it is designed for series that are centered around zero, like that in Figure 12.4(a). The test equation is

$$\Delta y_t = \gamma y_{t-1} + \sum_{s=1}^{p-1} a_s \Delta y_{t-s} + v_t \tag{12.27}$$

[6] The trend term takes the values 0, 1, 2, …,132 with 1984Q1 = 0.

TABLE 12.3 AR Processes and the Dickey–Fuller Tests

AR Processes: $\lvert\rho\rvert < 1$	Setting $\rho = 1$	Dickey–Fuller Tests
$y_t = \rho y_{t-1} + u_t$	$y_t = y_{t-1} + u_t$	Test with no constant and no trend
$y_t = \alpha + \rho y_{t-1} + v_t$ $\alpha = \mu(1 - \rho)$	$y_t = y_{t-1} + v_t$ $\alpha = 0$	Test with constant and no trend
$y_t = \alpha + \rho y_{t-1} + \lambda t + v_t$ $\alpha = \mu(1 - \rho) + \rho\delta$ $\lambda = \delta(1 - \rho)$	$y_t = \delta + y_{t-1} + v_t$ $\alpha = \delta$ $\lambda = 0$	Test with constant and trend

We test $H_0 : \gamma = 0$ against $H_1 : \gamma < 0$ as described previously, and the critical values are given in the first row of Table 12.2.

Most time series measured in terms of their original levels do not have a zero mean. However, their first differences $\Delta y_t = y_t - y_{t-1}$ may turn out to have a zero mean. For example, the first difference of the random walk $y_t = y_{t-1} + v_t$ is $\Delta y_t = v_t$ which has a zero mean. Testing whether first differences are stationary has relevance for finding the **order of integration** of a series which we consider in Section 12.3.6.

In Table 12.3, we summarize the models under H_0 and H_1 for each of the three tests, omitting the augmentation terms to avoid cluttering the table.

12.3.6 Order of Integration

Up to this stage, we have discussed only whether a series is stationary or nonstationary. We can take the analysis another step forward and consider a concept called the "order of integration." Recall that if y_t follows a random walk, then $\gamma = 0$ and the first difference of y_t becomes

$$\Delta y_t = y_t - y_{t-1} = v_t$$

An interesting feature of the series $\Delta y_t = y_t - y_{t-1}$ is that it is stationary since v_t, being an independent $\left(0, \sigma_v^2\right)$ random variable, is stationary. Series like y_t, which can be made stationary by taking the first difference, are said to be **integrated of order one**, and denoted as **I(1)**. Stationary series are said to be integrated of order zero, **I(0)**. In general, the order of integration of a series is the minimum number of times it must be differenced to make it stationary.

EXAMPLE 12.7 | The Order of Integration of the Two Interest Rate Series

In Example 12.4, we concluded that the two interest rate series *FFR* and *BR* were nonstationary. To find their order of integration, we ask the next question: are their first differences, $\Delta FFR_t = FFR_t - FFR_{t-1}$ and $\Delta BR_t = BR_t - BR_{t-1}$ stationary? Their plots, in Figures 12.1(f) and (h), suggest stationarity. Given these plots appear to fluctuate around zero, we use the Dickey–Fuller test equation with no intercept and no trend, to obtain the following results.

$$\widehat{\Delta(\Delta FFR_t)} = -0.715\Delta FFR_{t-1} + 0.157\Delta(\Delta FFR_{t-1})$$
$$(\tau \text{ and } t) \quad (-17.76) \qquad (4.33)$$

$$\widehat{\Delta(\Delta BR_t)} = -0.811\Delta BR_{t-1} + 0.235\Delta(\Delta BR_{t-1})$$
$$(\tau \text{ and } t) \quad (-19.84) \qquad (6.58)$$

where $\Delta(\Delta FFR_t) = \Delta FFR_t - \Delta FFR_{t-1}$ and $\Delta(\Delta BR_t) = \Delta BR_t - \Delta BR_{t-1}$. In both cases, one augmentation term was sufficient to eliminate serial correlation in the errors. Note that the null hypotheses are that the variables ΔF and ΔB are not stationary. The large negative values of the τ-statistic, $\tau = -17.76$ for ΔFFR and $\tau = -19.84$ for ΔBR, are much less and the 5% critical value $\tau_{0.05} = -1.94$. We therefore reject null hypotheses that ΔFFR and ΔBR have unit roots and conclude they are stationary.

These results imply that, while the levels of the two interest rates are nonstationary, their first differences are stationary. We say that the series FFR_t and BR_t are I(1) because they had to be differenced once to make them stationary $[\Delta FFR_t$ and ΔBR_t are I(0)$]$. In the Sections 12.4 and 12.5, we investigate the implications of these results for regression modeling.

12.3.7 Other Unit Root Tests

While augmented Dickey–Fuller tests remain the most popular tests for unit roots, the power of the tests is low in the sense that they often cannot distinguish between a highly persistent stationary process (where ρ is very close but not equal to 1) and a nonstationary process (where $\rho = 1$). The power of the test also diminishes as deterministic terms constant and trend are included in the test equation. Here we briefly mention other tests that have been developed with a view to improving the power of the test: the Elliot, Rothenberg, and Stock (ERS), Phillips and Perron (PP), Kwiatkowski, Phillips, Schmidt, and Shin (KPSS), and Ng and Perron (NP) tests.[7] Each test carries an abbreviation from the names of its developers.

The ERS test proposes removing the constant/trend effects from the data and performing the unit root test on the residuals. The distribution of the *t*-statistic is now devoid of deterministic terms (i.e., the constant and/or trend). The PP test adopts a nonparametric approach that assumes a general autoregressive moving-average structure and uses spectral methods to estimate the standard error of the test correlation. Instead of specifying a null hypothesis of nonstationary, the KPSS test specifies a null hypothesis that the series is stationary or trend stationary. NP tests suggest various modifications of the PP and ERS tests.

12.4 Cointegration

As a general rule, to avoid the problem of spurious regression, nonstationary time-series variables should not be used in regression models. However, there is an exception to this rule. If y_t and x_t are nonstationary I(1) variables, then we expect their difference, or any linear combination of them, such as $e_t = y_t - \beta_1 - \beta_2 x_t$,[8] to be I(1) as well. However, there is an important case when $e_t = y_t - \beta_1 - \beta_2 x_t$ is a stationary I(0) process. In this case, y_t and x_t are said to be **cointegrated**. Cointegration implies that y_t and x_t share similar stochastic trends, and, since the difference e_t is stationary, they never diverge too far from each other.

A natural way to test whether y_t and x_t are cointegrated is to test whether the errors $e_t = y_t - \beta_1 - \beta_2 x_t$ are stationary. Since we cannot observe e_t, we test the stationarity of the OLS residuals, $\hat{e}_t = y_t - b_1 - b_2 x_t$ using a Dickey–Fuller test. The test for cointegration is effectively a test of the stationarity of the residuals. If the residuals are stationary, then y_t and x_t are said to be cointegrated; if the residuals are nonstationary, then y_t and x_t are not cointegrated, and any apparent regression relationship between them is said to be spurious.

The test for stationarity of the residuals is based on the test equation

$$\Delta\hat{e}_t = \gamma\hat{e}_{t-1} + v_t \tag{12.28}$$

where $\Delta\hat{e}_t = \hat{e}_t - \hat{e}_{t-1}$. As before, we examine the *t* (or *tau*) statistic for the estimated slope coefficient. Note that the regression has no constant term because the mean of the regression residuals

[7] More details can be found in William Greene, *Econometric Analysis*, 8th ed., Chapter 21, 2018, Pearson.

[8] A linear combination of x and y is a new variable $z = a_0 + a_1 x + a_2 y$. Here we set the constants $a_0 = -\beta_1$, $a_1 = -\beta_2$, and $a_2 = 1$ and call z the series e.

TABLE 12.4 Critical Values for the Cointegration Test

Regression Model	1%	5%	10%
(1) $y_t = \beta x_t + e_t$	−3.39	−2.76	−2.45
(2) $y_t = \beta_1 + \beta_2 x_t + e_t$	−3.96	−3.37	−3.07
(3) $y_t = \beta_1 + \delta t + \beta_2 x_t + e_t$	−3.98	−3.42	−3.13

Note: These critical values are taken from J. Hamilton, *Time Series Analysis*, Princeton University Press, 1994, p. 766.

is zero. Also, since we are basing this test upon **estimated** values of the residuals, the critical values will be different from those in Table 12.2. The proper critical values for a test of cointegration are given in Table 12.4. The test equation can also include extra terms like $\Delta\hat{e}_{t-1}, \Delta\hat{e}_{t-2}, \ldots$ on the right-hand side if they are needed to eliminate autocorrelation in v_t.

There are three sets of critical values. Which set we use depends on whether the residuals $\hat{e}_t$ are derived from a regression equation without a constant term [like (12.29a)] or a regression equation with a constant term [like (12.29b)], or a regression equation with a constant and a time trend [like (12.29c)].

$$\text{Equation 1}: \hat{e}_t = y_t - bx_t \tag{12.29a}$$

$$\text{Equation 2}: \hat{e}_t = y_t - b_2 x_t - b_1 \tag{12.29b}$$

$$\text{Equation 3}: \hat{e}_t = y_t - b_2 x_t - b_1 - \hat{\delta} t \tag{12.29c}$$

EXAMPLE 12.8 | Are the Federal Funds Rate and Bond Rate Cointegrated?

To illustrate, let us test whether $y_t = BR_t$ and $x_t = FFR_t$, as plotted in Figures 12.1(e) and (g), are cointegrated. We have already shown that both series are nonstationary. The estimated least-squares regression between these variables is

$$\widehat{BR}_t = 1.328 + 0.832\, FFR_t \qquad R^2 = 0.908$$
$$(t) \qquad\qquad (85.72) \tag{12.30}$$

The estimated test equation for stationarity in the OLS residuals $\hat{e}_t = BR_t - 1.328 - 0.832\, FFR_t$ is

$$\widehat{\Delta\hat{e}_t} = -0.0817\, \hat{e}_{t-1} + 0.223\, \Delta\hat{e}_{t-1} - 0.177\, \Delta\hat{e}_{t-2}$$
$$(\tau \text{ and } t) \quad (-5.53) \qquad (6.29) \qquad (-4.90)$$

Note that this is the augmented Dickey–Fuller version of the test with two lagged terms Δe_{t-1} and Δe_{t-2} to correct for autocorrelation. Since there is a constant term in (12.30), we use the equation (2) critical values in Table 12.4.

The null and alternative hypotheses in the test for cointegration are

$$H_0: \text{the series are not cointegrated}$$
$$\Longleftrightarrow \text{residuals are nonstationary}$$
$$H_1: \text{the series are cointegrated}$$
$$\Longleftrightarrow \text{residuals are stationary}$$

Similar to the one-tail unit root tests, we reject the null hypothesis of no cointegration if $\tau \leq \tau_c$, and we do not reject the null hypothesis that the series are not cointegrated if $\tau > \tau_c$. The tau statistic in this case is −5.53 which is less than the critical value −3.37 at the 5% level of significance. Thus, we reject the null hypothesis that the least-squares residuals are nonstationary and conclude that they are stationary. This implies that the bond rate and the federal funds rate are cointegrated. In other words, there is a fundamental relationship between these two variables (the estimated regression relationship between them is valid and not spurious) and the estimated values of the intercept and slope are 1.328 and 0.832, respectively.

The result—that the federal funds and bond rates are cointegrated—has major economic implications! It means that when the Federal Reserve implements monetary policy by changing the federal funds rate, the bond rate will also change thereby ensuring that the effects of monetary policy are transmitted to the rest of the economy. In contrast, the effectiveness of monetary policy would be severely hampered if the bond and federal funds rates were spuriously related as this implies that their movements, fundamentally, have little to do with each other.

12.4.1 The Error Correction Model

In Section 12.4, we discussed the concept of cointegration as the relationship between I(1) variables such that the residuals are I(0). A relationship between I(1) variables is also often referred to as a long-run relationship while a relationship between I(0) variables is often referred to as a short-run relationship. In this section, we describe a dynamic relationship between I(0) variables, which embeds a cointegrating relationship, known as the short-run error correction model.

As discussed in Chapter 9, when one is working with time-series data, it is quite common, and in fact, quite important to allow for dynamic effects. To derive the error correction model requires a bit of algebra, but we shall persevere as this model offers a coherent way to combine the long- and short-run effects.

Let us start with a general model that contains lags of y and x, namely the ARDL model introduced in Chapter 9, except that now the variables are nonstationary:

$$y_t = \delta + \theta_1 y_{t-1} + \delta_0 x_t + \delta_1 x_{t-1} + v_t$$

For simplicity, we shall consider lags up to order one, but the following analysis holds for any order of lags. Now recognize that if y and x are cointegrated, it means that there is a long-run relationship between them. To derive this exact relationship, we set $y_t = y_{t-1} = y$, $x_t = x_{t-1} = x$ and $v_t = 0$ and then, imposing this concept in the ARDL, we obtain

$$y(1 - \theta_1) = \delta + (\delta_0 + \delta_1)\, x$$

This equation can be rewritten as $y = \beta_1 + \beta_2 x$ where $\beta_1 = \delta/(1 - \theta_1)$ and $\beta_2 = (\delta_0 + \delta_1)/(1 - \theta_1)$. To repeat, we have now derived the implied cointegrating relationship between y and x; alternatively, we have derived the long-run relationship that holds between the two I(1) variables.

We will now manipulate the ARDL to see how it embeds the cointegrating relation. First, add the term $-y_{t-1}$ to both sides of the equation:

$$y_t - y_{t-1} = \delta + (\theta_1 - 1)\, y_{t-1} + \delta_0 x_t + \delta_1 x_{t-1} + v_t$$

Second, add the term $-\delta_0 x_{t-1} + \delta_0 x_{t-1}$ to the right-hand side to obtain

$$\Delta y_t = \delta + (\theta_1 - 1) y_{t-1} + \delta_0 (x_t - x_{t-1}) + (\delta_0 + \delta_1) x_{t-1} + v_t$$

where $\Delta y_t = y_t - y_{t-1}$. If we then manipulate the equation to look like

$$\Delta y_t = (\theta_1 - 1)\left(\frac{\delta}{(\theta_1 - 1)} + y_{t-1} + \frac{(\delta_0 + \delta_1)}{(\theta_1 - 1)} x_{t-1}\right) + \delta_0 \Delta x_t + v_t$$

where $\Delta x_t = x_t - x_{t-1}$, and do a bit more tidying, using the definitions β_1 and β_2, we get

$$\Delta y_t = -\alpha(y_{t-1} - \beta_1 - \beta_2 x_{t-1}) + \delta_0 \Delta x_t + v_t \tag{12.31}$$

where $\alpha = (1 - \theta_1)$. As you can see, the expression in parenthesis is the cointegrating relationship. In other words, we have embedded the cointegrating relationship between y and x in a general ARDL framework.

Equation (12.31) is called an error correction equation because (a) the expression $(y_{t-1} - \beta_1 - \beta_2 x_{t-1})$ shows the deviation of y_{t-1} from its long-run value, $\beta_1 + \beta_2 x_{t-1}$—in other words, the "error" in the previous period—and (b) the term $(\theta_1 - 1)$ shows the "correction" of Δy_t to the "error." More specifically, if the error in the previous period is positive so that $y_{t-1} > (\beta_1 + \beta_2 x_{t-1})$, then y_t should fall and Δy_t should be negative; conversely, if the error in the previous period is negative so that $y_{t-1} < (\beta_1 + \beta_2 x_{t-1})$, then y_t should rise and Δy_t should be positive. This means that if a cointegrating relationship between y and x exists, so that adjustments always work to "error-correct," then empirically we should also find that $(1 - \theta_1) > 0$, which implies that $\theta_1 < 1$. If there is no evidence of cointegration between the variables, then the estimate for θ_1 would be insignificant.

The error correction model is a very popular model because it allows for the existence of an underlying or fundamental link between variables (the long-run relationship) as well as for short-run adjustments (i.e., changes) between variables, including adjustments toward the cointegrating relationship. It also shows that we can work with I(1) variables (y_{t-1}, x_{t-1}) and I(0) variables $(\Delta y_t, \Delta x_t)$ in the same equation provided that (y, x) are cointegrated, meaning that the term $(y_{t-1} - \beta_0 - \beta_1 x_{t-1})$ contains stationary residuals. In fact, this formulation can also be used to test for cointegration between y and x.

To estimate (12.31) we can proceed in one of two ways: we can estimate the equation with $y_{t-1} - \beta_1 - \beta_2 x_{t-1}$ replaced by $\hat{e}_{t-1}$, or we can find new estimates of β_1 and β_2 at the same time as we estimate α and δ_0. For the latter approach, we can estimate the parameters directly by applying nonlinear least squares to (12.31), or we can use OLS to estimate the equation

$$\Delta y_t = \beta_1^* + \alpha^* y_{t-1} + \beta_2^* x_{t-1} + \delta_0 \Delta x_{t-1} + v_t$$

and retrieve the parameters in equation (12.31) from $\alpha = -\alpha^*$, $\beta_1 = -\beta_1^*/\alpha^*$ and $\beta_2 = -\beta_2^*/\alpha^*$. The nonlinear least squares and the retrieved OLS estimates will be identical. However, they will differ slightly from the two-step estimates obtained by replacing $y_{t-1} - \beta_1 - \beta_2 x_{t-1}$ with $\hat{e}_{t-1}$.

EXAMPLE 12.9 | An Error Correction Model for the Bond and Federal Funds Rates

For an error correction model relating changes in the bond rate to the lagged cointegrating relationship and changes in the federal funds rate, it turns out that up to four lags of ΔFFR_t are relevant and two lags of ΔBR_t are needed to eliminate serial correlation in the error. The equation estimated directly using nonlinear least squares is

$$\begin{aligned}
\widehat{\Delta BR_t} = {} & \underset{(3.90)}{-0.0464}(BR_{t-1} - 1.323 - 0.833FFR_{t-1}) \quad (t)\\
& + \underset{(7.27)}{0.272}\Delta BR_{t-1} - \underset{(-6.40)}{0.242}\Delta BR_{t-2}\\
& + \underset{(14.22)}{0.342}\Delta FFR_t - \underset{(-3.83)}{0.105}\Delta FFR_{t-1} + \underset{(3.62)}{0.099}\Delta FFR_{t-2}\\
& - \underset{(-2.69)}{0.066}\Delta FFR_{t-3} + \underset{(2.46)}{0.056}\Delta FFR_{t-4}
\end{aligned} \qquad (12.32)$$

Notice that the estimates $\hat{\beta}_1 = 1.323$ and $\hat{\beta}_2 = 0.833$ are very similar to those obtained from direct OLS estimation of the cointegrating relationship in (12.30). The relationship between all the coefficients in (12.32) and its corresponding ARDL model are explored in Exercise 12.18.

If we use the residuals $\hat{e}_t = BR_t - 1.323 - 0.833FFR_t$, obtained from the estimates in (12.32), to test for cointegration, we get a similar result to our earlier one

$$\widehat{\Delta e_t} = \underset{(-5.53)}{-0.0819}\hat{e}_{t-1} + \underset{(6.29)}{0.224}\Delta\hat{e}_{t-1} - \underset{(-4.90)}{0.177}\Delta\hat{e}_{t-2} \quad (\tau \text{ and } t)$$

As before, the null hypothesis is that (BR, FFR) are not cointegrated (the residuals are nonstationary). Since the cointegrating relationship includes a constant, the critical value from Table 12.4 is −3.37. Comparing the actual value $\tau = -5.53$ with the critical value, we reject the null hypothesis and conclude that (BR, FFR) are cointegrated.

12.5 Regression When There Is No Cointegration

Thus far, we have shown that regression with I(1) variables is acceptable providing those variables are cointegrated, allowing us to avoid the problem of spurious results. We also know that regression with stationary I(0) variables, that we studied in Chapter 9, is acceptable. What happens when there is no cointegration between I(1) variables? In this case, the sensible thing to do is to convert the nonstationary series to stationary series and to use the techniques discussed in Chapter 9 to estimate dynamic relationships between the stationary variables. However, we stress that this step should be taken only when we fail to find cointegration between the I(1) variables.

Regression with cointegrated I(1) variables makes the least-squares estimator "super-consistent"[9] and, moreover, it is economically useful to establish relationships between the levels of economic variables.

How we convert nonstationary series to stationary series, and the kind of model we estimate, depend on whether the variables are **difference stationary** or **trend stationary**. In the former case, we convert the nonstationary series to its stationary counterpart by taking first differences. We dealt with the latter case in Section 12.1.1 where we converted the nonstationary series to a stationary series by detrending, or we included a trend term in the regression relationship. We now consider how to estimate regression relationships with nonstationary variables that are neither cointegrated nor trend stationary.

Recall that an I(1) variable is one that is stationary after differencing once. Another name for variables with this characteristic is that they are **first-difference stationary**. Specifically, if y_t is nonstationary with a stochastic trend and its first difference $\Delta y_t = y_t - y_{t-1}$ is stationary, then y_t is I(1) and first-difference stationary. If Dickey–Fuller tests reveal that two variables, y and x, that you would like to relate in a regression, are first difference stationary and not cointegrated, then a suitable regression involving only stationary variables is one that relates changes in y to changes in x, with relevant lags included. If y_t and x_t behave like random walks with no obvious trend, then an intercept can be omitted. For example, using one lagged Δy_t and a current and lagged Δx_t, we have:

$$\Delta y_t = \theta \Delta y_{t-1} + \beta_0 \Delta x_t + \beta_1 \Delta x_{t-1} + e_t \tag{12.33}$$

If y_t and x_t behave like random walks with drift, then it is appropriate to include an intercept, an example of which is

$$\Delta y_t = \alpha + \theta \Delta y_{t-1} + \beta_0 \Delta x_t + \beta_1 \Delta x_{t-1} + e_t \tag{12.34}$$

Note that a random walk with drift is such that $\Delta y_t = \alpha + v_t$, implying an intercept should be included, whereas a random walk with no drift becomes $\Delta y_t = v_t$. In line with Chapter 9, the models in (12.33) and (12.34) are ARDL models with first-differenced variables. In general, since there is often doubt about the role of the constant term, the usual practice is to include an intercept term in the regression.

EXAMPLE 12.10 | A Consumption Function in First Differences

In Chapter 9, there were a number of examples and exercises involving first differences of variables. When studying that chapter, you may have wondered why we did not use variables in their levels. The reason is now clear. It was to ensure the variables were stationary. In the following example of a consumption function, we return to the data file *cons_inc*, containing quarterly data on Australian consumption expenditure and national disposable income, used earlier in Example 9.16. We will use data from 1985Q1 to 2016Q3. Plots of the series appear in Figure 12.6.

Since both consumption (C) and income (Y) are clearly trending, we include a trend term in the Dickey–Fuller test equations to see if they should be treated as trend stationary or difference stationary. The results from the test equations are

$$\widehat{\Delta C_t} = 1989.7 + 29.43\,t - 0.0193 C_{t-1} + 0.244 \Delta C_{t-1}$$
$$(\tau \text{ and } t) \qquad (2.03)\ (-1.70) \qquad (2.82)$$

$$\widehat{\Delta Y_t} = 5044.6 + 80.04\,t - 0.0409 Y_{t-1} + 0.248 \Delta Y_{t-1}$$
$$(\tau \text{ and } t) \qquad (2.27)\ (-2.14) \qquad (2.89)$$

From Table 12.2, the 5% critical value for test equations that include a trend is $\tau_{0.05} = -3.41$. The τ values for consumption (−1.70) and income (−2.14) are both greater than $\tau_{0.05}$. Hence, we are unable to conclude that C and Y are trend stationary.

[9]Consistency means that as $T \to \infty$ the least squares estimator converges to the true parameter value. See Section 5.7. Super-consistency means that it converges to the true value at a faster rate.

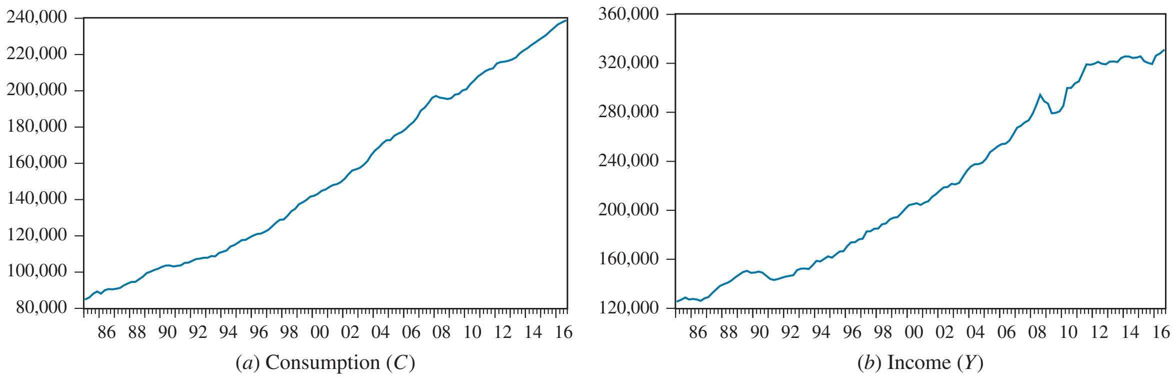

FIGURE 12.6 **Australian consumption and disposable income.**

The next step is to see if C and Y are cointegrated. Because they are trending, we include a trend term, and estimate the following equation, saving the residuals.

$$\hat{C}_t = -18746 + 420.4t + 0.468Y_t$$
$$(t) \qquad (9.92) \quad (20.49) \qquad (12.35)$$

If the residuals are stationary, we conclude C and Y are cointegrated and (12.35) is a valid regression. If the residuals are nonstationary, then (12.35) could be a spurious regression. The test equation for assessing the stationarity of the residuals is

$$\widehat{\Delta\hat{e}_t} = -0.121\hat{e}_{t-1} + 0.263\Delta\hat{e}_{t-1}$$
$$(\tau \text{ and } t) \quad (-2.93) \qquad (2.94)$$

Comparing $\tau = -2.93$ with the critical value of $\tau_{0.05} = -3.42$ in the third row of Table 12.4, we fail to reject a null hypothesis that the residuals are nonstationary (C and Y are not cointegrated).

Having established that C and Y are not trend stationary and not cointegrated, or at least that there is insufficient evidence to suggest otherwise, the natural regression to estimate relating the two variables is one in first differences. First, however, we need to confirm that they are first-difference stationary (integrated of order one). The unit-root test equations for this purpose are

$$\widehat{\Delta(\Delta C_t)} = 844.0 - 0.689\Delta C_{t-1}$$
$$(\tau) \qquad (-8.14)$$

$$\widehat{\Delta(\Delta Y_t)} = 1{,}228.7 - 0.751\Delta Y_{t-1}$$
$$(\tau) \qquad (-8.68)$$

We include a constant in these equations because the unit-root test for the variables in their levels included a trend. The test values $\tau = -8.14$ and $\tau = -8.68$ are less than 5% critical value $\tau_{0.05} = -2.86$ from Table 12.2. We therefore conclude that ΔC and ΔY are stationary and hence that C and Y are first-difference stationary. Proceeding to estimate an ARDL model for C and Y in first differences, we obtain

$$\widehat{\Delta C_t} = 785.8 + 0.0573\Delta Y_t + 0.282\Delta C_{t-1}$$
$$(t) \qquad (2.07) \qquad (3.34)$$

12.6 Summary

- If variables are stationary, or I(1) and cointegrated, we can estimate a regression relationship between the levels of those variables without fear of encountering a spurious regression. In the latter case, we can do this by estimating a least-squares equation between the I(1) variables or by estimating a nonlinear least-squares error correction model which embeds the I(1) variables.
- If the variables are I(1) and not cointegrated, we need to estimate a relationship in first differences, with or without the constant term.

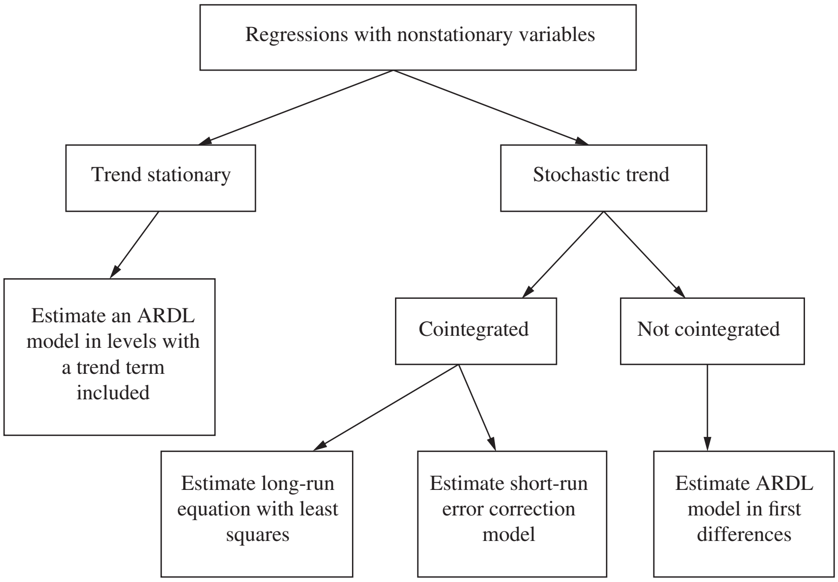

FIGURE 12.7 **Regression with time-series data: nonstationary variables.**

- If they are trend stationary, we can either detrend the series first and then perform regression analysis with the stationary (detrended) variables or, alternatively, estimate a regression relationship that includes a trend variable.

These options are shown in Figure 12.7.

12.7 Exercises

12.7.1 Problems

12.1 Consider the AR(2) model $y_t = \delta + \theta_1 y_{t-1} + \theta_2 y_{t-2} + v_t$. Suppose that

$$1 - \theta_1 z - \theta_2 z^2 = (1 - c_1 z)(1 - c_2 z)$$

a. Show that $c_1 + c_2 = \theta_1$ and $c_1 c_2 = -\theta_2$.

b. Prove that the AR(2) model has a unit root if and only if $\theta_1 + \theta_2 - 1 = 0$. [*Hint*: The roots of $1 - \theta_1 z - \theta_2 z^2 = 0$ are $1/c_1$ and $1/c_2$.]

c. Prove that $\theta_1 + \theta_2 - 1 < 0$ if the AR(2) process is stationary.

d. Prove that the AR(2) model $y_t = \delta + \theta_1 y_{t-1} + \theta_2 y_{t-2} + v_t$ can also be written as

$$\Delta y_t = \delta + \gamma y_{t-1} + a_1 \Delta y_{t-1} + v_t$$

where $\gamma = \theta_1 + \theta_2 - 1$ and $a_1 = -\theta_2$. What are the implications of this result and the results in parts (b) and (c) for unit root tests in an AR(2) model.

e. Show that an AR(p) model has a unit root if $\gamma = \theta_1 + \theta_2 + \cdots + \theta_p - 1 = 0$.

f. Show that setting $\gamma = \theta_1 + \theta_2 + \cdots + \theta_p - 1$ in equation (12.23) implies $a_j = -\sum_{r=j}^{p-1} \theta_{r+1}$.

12.2 **a.** Consider the stationary AR(1) model $y_t = \rho y_{t-1} + v_t$, $|\rho| < 1$. The v_t are independent random errors with mean zero and variance σ_v^2. In Appendix 9B we showed that the autocorrelations for this model are given by $\text{corr}(y_t, y_{t+s}) = \rho^s$. Given $\rho = 0.9$, find the autocorrelations for observations 1 period apart, 2 periods apart, etc., up to 10 periods apart.

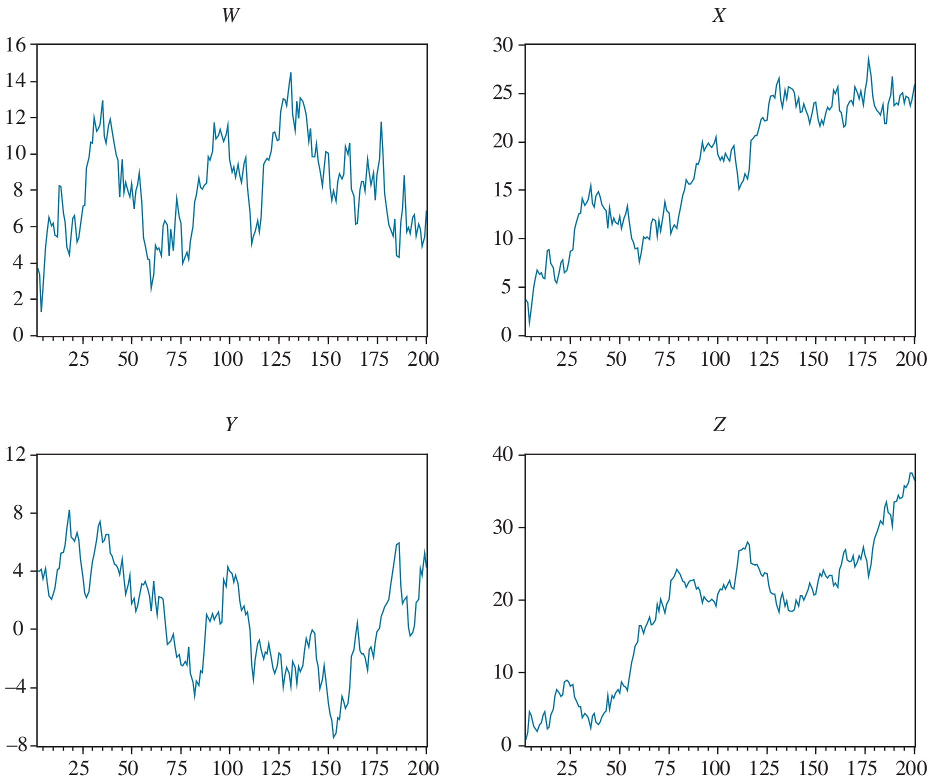

FIGURE 12.8 **Time series for Exercise 12.3.**

b. Consider the nonstationary random walk model $y_t = y_{t-1} + v_t$. Assuming a fixed $y_0 = 0$, rewrite y_t as a function of all past errors $v_{t-1}, v_{t-2}, \ldots, v_1$.

c. Use the result in part (b) to find (i) the mean of y_t, (ii) the variance of y_t, and (iii) the covariance between y_t and y_{t+s}.

d. Use the results from part (c) to show that $\text{corr}\left(y_t, y_{t+s}\right) = \sqrt{t/(t+s)}$.

e. Assume $t = 100$ (the random walk has been operating for 100 periods). Find the correlations between y_{100} and y in each of the next 10 periods (up to y_{110}). Compare these correlations with those obtained in part (a).

f. Find $\text{corr}(y_{100}, y_{200})$ for each of the two models and comment on their magnitudes.

12.3 Figure 12.8 shows plots of four time series that are stored in the data file *unit*.

a. The results from Dickey–Fuller test equations for these four variables are given below. Explain why these equations were chosen. No augmentation terms are included. What criteria would have led to their omission?

$$\widehat{\Delta W_t} = 0.778 - 0.0936 W_{t-1}$$
$$(\tau) \qquad (-3.23)$$

$$\widehat{\Delta Y_t} = 0.0304 - 0.0396 Y_{t-1}$$
$$(\tau) \qquad (-1.98)$$

$$\widehat{\Delta X_t} = 0.805 - 0.0939 X_{t-1} + 0.00928\, t$$
$$(\tau) \qquad (-3.13)$$

$$\widehat{\Delta Z_t} = 0.318 - 0.0355 Z_{t-1} + 0.00306\, t$$
$$(\tau) \qquad (-1.87)$$

b. Carry out Dickey–Fuller tests on each of the series. What do you conclude about their stationarity properties?

c. The following estimated equation uses residuals $\hat{e}_t$ from least-squares estimation of the model $X_t = \beta_1 + \delta t + \beta_2 Z_t + e_t$. Can you conclude that X_t and Z_t are cointegrated?

$$\widehat{\Delta \hat{e}_t} = -0.0683 \hat{e}_{t-1}$$
$$(\tau) \quad (-2.83)$$

d. Use the following equation and earlier results to assess the order of integration of Z_t.

$$\widehat{\Delta(\Delta Z_t)} = 0.174 - 0.987 \Delta Z_{t-1}$$
$$(\tau) \qquad (-13.76)$$

12.4 A time-series process of the form $y_t = \alpha + y_{t-1} + v_t$, $v_t \sim N(0, \sigma^2)$ can be rearranged as $y_t - y_{t-1} = \Delta y_t = \alpha + v_t$. This shows that y_t is integrated of order one, since its first difference is stationary. Show that a time series of the form $y_t = 2y_{t-1} - y_{t-2} + \alpha + v_t$ is integrated of order two.

12.5 In Chapter 9, we found that, given a time series of observations, $I_T = \{(y_1, x_1), (y_2, x_2), \ldots, (y_T, x_T)\}$, the best one-period and two-period ahead forecasts for y_{T+1} and y_{T+2} were given by $E(y_{T+1}|I_T)$ and $E(y_{T+2}|I_T)$, respectively. Given that $T = 29$, $y_T = 10$, $y_{T-1} = 12$, $x_{T+2} = x_{T+1} = x_T = 5$, and $x_{T-1} = 6$, find forecasts for y_{T+1} and y_{T+2} from each of the following models. In each case, assume that v_t are independent random errors distributed as $N(0, \sigma_v^2 = 4)$.

a. The random walk $y_t = y_{t-1} + v_t$.
b. The random walk with drift $y_t = 5 + y_{t-1} + v_t$.
c. The random walk $\ln(y_t) = \ln(y_{t-1}) + v_t$.
d. The deterministic trend model $y_t = 10 + 0.1\,t + v_t$.
e. The ARDL model $y_t = 6 + 0.6y_{t-1} + 0.3x_t + 0.1x_{t-1} + v_t$.
f. The error correction model $\Delta y_t = -0.4(y_{t-1} - 15 - x_{t-1}) + 0.3\Delta x_t + v_t$. In addition, find the long-run equilibrium value for y when $x = 5$.
g. The first difference model $\Delta y_t = 0.6\Delta y_{t-1} + 0.3\Delta x_t + 0.1\Delta x_{t-1} + v_t$.

12.6 Increases in the mortgage interest rate increase the cost of owning a house and lower the demand for houses. In this question we investigate the properties of two time series that could be used to model this demand relationship: the number of new one-family houses sold in the U.S. (*HOMES*) and the 30-year conventional mortgage rate (*IRATE*). These series, along with their changes, *DHOMES* and *DIRATE*, are plotted in Figure 12.9. The data are from January 1992 (1992M1) to September 2016 (2016M9). The units of measurement are thousands of new houses for *HOMES* and percentage points for *IRATE*.

Use the following test equation results to test for unit roots. In each case give the null and alternative hypotheses, and draw a conclusion. In all cases use a 5% significance level. Based on the test results, describe how you would set up a model for the demand relationship. (In each model, the OLS standard errors do not reflect the true variance of the estimator $\hat{\gamma}$ but, nevertheless, they can be used to construct the τ-statistic.)

a. $$\widehat{\Delta HOMES_t} = 7.051 - 0.0102 HOMES_{t-1} - 0.280 \Delta HOMES_{t-1}$$
$$(se) \qquad (0.0096) \qquad (0.056)$$

b. $$\widehat{\Delta HOMES_t} = 16.36 - 0.0385\,t - 0.0151 HOMES_{t-1} - 0.279 \Delta HOMES_{t-1}$$
$$(se) \qquad (0.0345) \quad (0.0106) \qquad (0.056)$$

c. $$\widehat{\Delta IRATE_t} = 0.0477 - 0.00985 IRATE_{t-1} + 0.300 \Delta IRATE_{t-1}$$
$$(se) \qquad (0.00679) \qquad (0.056)$$

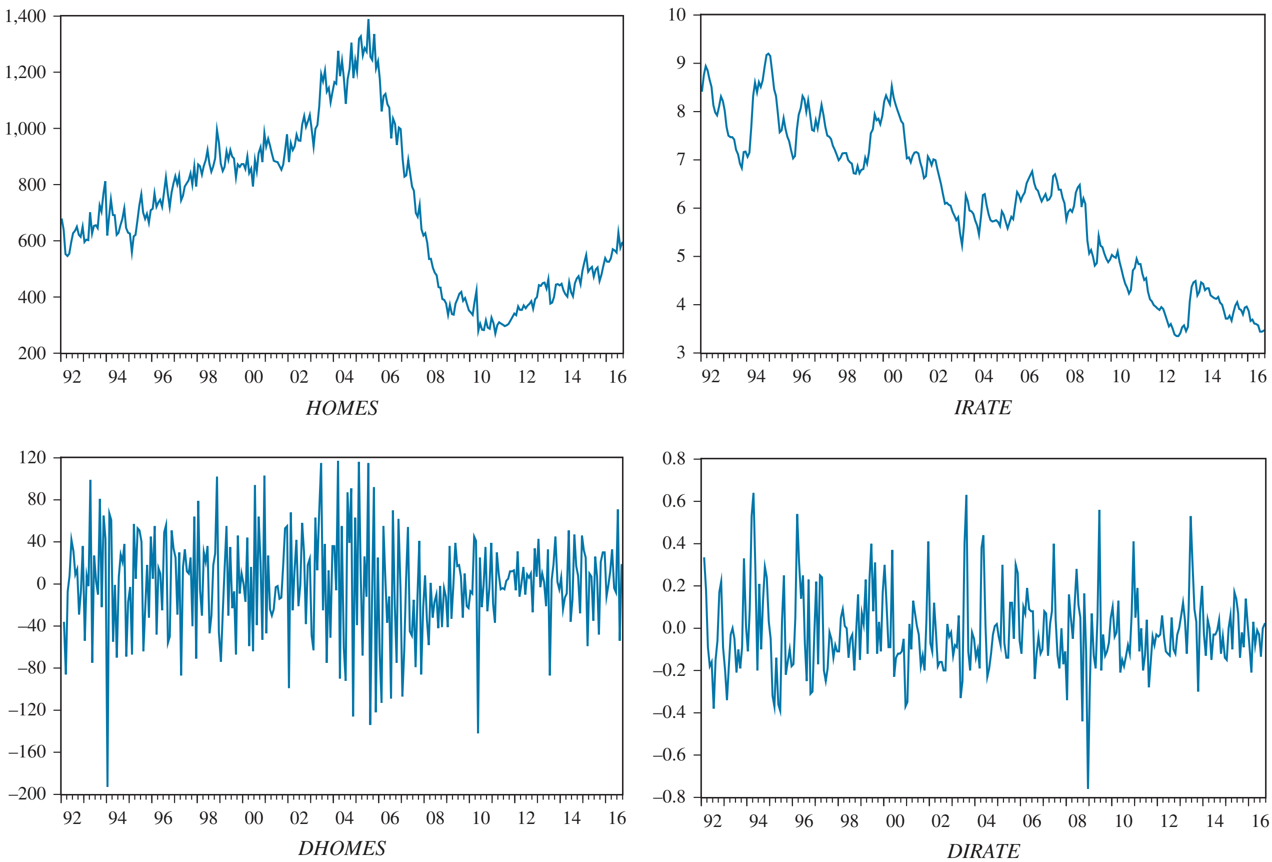

FIGURE 12.9 **Time series for new houses and the mortgage rate and their changes.**

d. $\widehat{\Delta IRATE_t} = 0.603 - 0.00120t - 0.0710 IRATE_{t-1} + 0.329 \Delta IRATE_{t-1}$

(se) (0.00033) (0.0181) (0.055)

e. $\widehat{\Delta DHOMES_t} = -0.254 - 1.285 DHOMES_{t-1}$

(se) (0.056)

f. $\widehat{\Delta DIRATE_t} = -0.0151 - 0.816 DIRATE_{t-1} + 0.151 \Delta DIRATE_{t-1}$

(se) (0.069) (0.058)

g. In the following test equation the $\hat{e}_t$ are the residuals from estimating the equation $HOMES_t = \beta_1 + \beta_2 IRATE_t + e_t$.

$$\widehat{\Delta \hat{e}_t} = -0.0191 \hat{e}_{t-1} - 0.181 \Delta \hat{e}_{t-1}$$

(se) (0.0117) (0.057)

h. In the following test equation the $\hat{u}_t$ are the residuals from estimating the equation $HOMES_t = \beta_1 + \delta_t + \beta_2 IRATE_t + u_t$.

$$\widehat{\Delta \hat{u}_t} = -0.0180 \hat{u}_{t-1} - 0.208 \Delta \hat{u}_{t-1}$$

(se) (0.0114) (0.057)

12.7.2 Computer Exercises

12.7 The data file *usmacro* contains quarterly observations on the U.S. unemployment rate (U), the U.S. GDP growth rate (G), and the U.S. inflation rate (INF) from 1948Q1 to 2016Q1. Plot these series and perform unit root tests on them to assess whether or not they are stationary. In your answer, justify your choice of a test equation, present the results from estimating that equation, state the null and alternative hypotheses, and draw a conclusion. Use a 5% significance level. What are the orders of integration of the three series?

12.8 The data file *okun5_aus* contains quarterly observations on the Australian unemployment rate (U), and the Australian GDP growth rate (G) from 1978Q2 to 2016Q2. Plot these series and perform unit root tests on them to assess whether or not they are stationary. In your answer, justify your choice of a test equation, present the results from estimating that equation, state the null and alternative hypotheses, and draw a conclusion. Use a 5% significance level. What are the orders of integration of the two series?

12.9 The data file *phillips5_aus* contains quarterly observations on the Australian unemployment rate (U), and the Australian inflation rate (INF) from 1987Q1 to 2016Q1. Plot these series and perform unit root tests on them to assess whether or not they are stationary. In your answer, justify your choice of a test equation, present the results from estimating that equation, state the null and alternative hypotheses, and draw a conclusion. Use a 5% significance level. What are the orders of integration of the two series?

12.10 The data file *oil5* contains quarterly observations on the price of oil from 1980Q1 to 2016Q1.

- **a.** Plot the observations.
- **b.** Using data from 1980Q1 to 2015Q2, test whether the series is stationary or nonstationary. What is its order of integration?
- **c.** Using information from part (b), the sample period 1980Q1 to 2015Q2, and other relevant criteria, specify and estimate an AR model for the price of oil.
- **d.** Use the model estimated in part (c) to forecast the price of oil for 2015Q3, 2015Q4, and 2016Q1.
- **e.** Find the percentage forecast errors for each of the forecasts made in part (d). Are your forecasts accurate?

12.11 The data file *freddie1* contains a monthly housing price index for the price of houses in Beckley, West Virginia ($BEKLY$), and the monthly value of Australian exports to China ($XCHINA$), from 1988M1 to 2015M12.

- **a.** Estimate the regression equation $XCHINA_t = \beta_1 + \beta_2\, BEKLY_t + e_t$ and comment on the results.
- **b.** Plot the series $BEKLY$, $XCHINA$, and $\ln(XCHINA)$ and describe the graphs. Do they provide any insights into the results from part (a)?
- **c.** Estimate the equation $\ln(XCHINA_t) = \beta_1 + \delta t + \beta_2\, BEKLY_t + e_t$ and comment on the results. Suggest a reason why $\ln(XCHINA)$ rather than $XCHINA$ was chosen as the left-hand-side variable.
- **d.** Do unit root tests suggest $\ln(XCHINA)$ and $BEKLY$ are stationary or trend stationary? Do the test results provide any insights into the results in part (c)?

12.12 The data file *freddie2* contains monthly housing price indices for the prices of houses in Champaign-Urbana, Illinois ($CHURB$), and Charlottesville, Virginia ($CHARV$) from 1982M1 to 2015M12.

- **a.** Plot the two series on the one graph and comment on the plots.
- **b.** Using a 5% significance level, test each of the two series for unit roots and find the order of integration of each series. Explain your choice of test equations. Are the series trend stationary? Are the series first-difference stationary? Are the series second-difference stationary?
- **c.** Using a 5% significance level, test whether $CHURB$ and $CHARV$ are cointegrated.
- **d.** Plot the first differences of the two series on the one graph and comment on the plots.
- **e.** Using a 5% significance level, test whether the first differences of $CHURB$ and $CHARV$ are cointegrated.

12.13 The data file *ozconfn* contains quarterly data on Australian real consumption expenditure ($CONS$) and real net national disposable income (INC) from 1975Q1 to 2010Q4.

- **a.** Create the series $LCONS = \ln(CONS)$ and plot the series $LCONS$ and INC. Comment on the graphs.
- **b.** Detrend each of the series by estimating the linear trends $LCONS_t = a_1 + a_2 t + u_{1t}$ and $INC_t = c_1 + c_2 t + u_{2t}$, and saving the residuals. Use values $t = 0, 1, \ldots, T-1$ for the trend term.

c. Plot the detrended series and comment on the graphs.

d. From part (c), you will have noticed that there is a strong seasonal component in each series. Econometricians have developed several methods for removing a seasonal component or "seasonally adjusting" the data. One very simple method is to subtract out the effect of seasonal dummy variables. To use this method, and remove the trend at the same time, we estimate the equation

$$y_t = \pi_0 t + \pi_1 D_{1t} + \pi_2 D_{2t} + \pi_3 D_{3t} + \pi_4 D_{4t} + u_t \quad \text{(XR12.13)}$$

where $D_{jt} = 1$ when t is an observation in quarter j, and 0 otherwise. Estimate (XR12.13) for both *LCONS* and *INC* and save the residuals; call them $LCONS^*$ and INC^*.

e. Plot $LCONS^*$ and INC^* and compare these graphs with those obtained in part (c).

f. Using a 5% significance level and the critical values in the third row of Table 12.2, test whether $LCONS^*$ and INC^* are stationary or first-difference stationary. Explain your choice of test equation, and comment on the suitability of the critical values.

g. Estimate the following two equations and compare the estimates

$$LCONS_t = \delta t + \phi_1 D_{1t} + \phi_2 D_{2t} + \phi_3 D_{3t} + \phi_4 D_{4t} + \beta INC_t + e_t$$

$$LCONS_t^* = \beta INC_t^* + e_t$$

h. Using a 5% significance level, test whether the equation in part (g)—either equation—is a cointegrating relationship. What critical value did you use?

i. Estimate an error correction model relating $\Delta LCONS_t$ to ΔINC_t and, if relevant, the lagged cointegrating residuals from part (g).

12.14 The data file *gdp5* contains the data on *GDP* displayed in Figure 12.1.

a. Is *GDP* stationary or nonstationary? Explain your choice of test equation.

b. What is the order of integration of *GDP*?

c. Construct and estimate a suitable model for forecasting *GDP* in 2017Q1. What is your forecast?

12.15 The data file *usdata5* contains the data on inflation displayed in Figure 12.1.

a. Is inflation stationary or nonstationary? Explain your choice of test equation.

b. What is the order of integration of inflation?

c. Construct and estimate a suitable model for forecasting inflation in 2017M1. What is your forecast?

12.16 In Example 12.2, using data from the data file *toody5*, we estimated the model

$$\ln(YIELD_t) = \alpha_1 + \alpha_2 t + \beta_1 RAIN_t + \beta_2 RAIN_t^2 + e_t$$

An assumption underlying this example was that ln(*YIELD*), *RAIN*, and $RAIN^2$ are all trend stationary. Test this assumption using a 5% significance level.

12.17 a. Using data from the data file *toody5*, estimate the following model. Comment on the results.

$$YIELD_t = \alpha_1 + \alpha_2 t + \beta_1 RAIN_t + \beta_2 RAIN_t^2 + e_t$$

b. Plot the residuals from the model estimated in part (a) and check the residual correlogram. What do you observe?

c. Estimate the following model and comment on the results.

$$YIELD_t = \alpha_1 + \alpha_2 t + \alpha_3 t^2 + \beta_1 RAIN_t + \beta_2 RAIN_t^2 + e_t$$

d. Plot the residuals from the model estimated in part (c) and check the residual correlogram. How do the properties of the residuals differ from those in part (b)?

e. Using a 5% significance level, test whether *YIELD*, *RAIN*, and $RAIN^2$ are trend stationary after subtracting out the quadratic trend.

12.18 Consider the ARDL model

$$y_t = \delta + \sum_{s=1}^{3} \theta_s y_{t-s} + \sum_{r=0}^{5} \delta_r x_{t-r} + v_t \quad \text{(XR12.18)}$$

Assume that y_t and x_t are I(1) and cointegrated. Let the cointegrating relationship be described by the equation $y_t = \beta_1 + \beta_2 x_t + e_t$.

a. Show that $\beta_1 = \delta/(1 - \theta_1 - \theta_2 - \theta_3)$ and $\beta_2 = \sum_{r=0}^{5}\delta_r/(1 - \theta_1 - \theta_2 - \theta_3)$.

b. Consider the corresponding error correction model

$$\Delta y_t = -\alpha(y_{t-1} - \beta_1 - \beta_2 x_{t-1}) + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \sum_{r=0}^{4} \eta_r \Delta x_{t-r} + v_t$$

Show that $\delta = \alpha\beta_1$, $\theta_1 = 1 - \alpha + \phi_1$, $\theta_2 = \phi_2 - \phi_1$, $\theta_3 = -\phi_2$, $\delta_0 = \eta_0$, $\delta_1 = \alpha\beta_2 - \eta_0 + \eta_1$, $\delta_2 = \eta_2 - \eta_1$, $\delta_3 = \eta_3 - \eta_2$, $\delta_4 = \eta_4 - \eta_3$, and $\delta_5 = -\eta_4$.

c. Using the data in *usdata5*, set $y_t = BR_t$ and $x_t = FFR_t$ and find least-squares estimates of the parameters in equation (XR12.18).

d. Use nonlinear least squares to estimate equation (12.32) in Example 12.9.

e. Substitute the parameter estimates of equation (12.32) obtained in part (d) into the expressions given in part (b) and compare the estimates you get with those obtained in part (c). What conclusion can you draw from this comparison?

12.19 When we estimated an error correction model for the bond and federal funds rates in Example 12.9, we estimated the coefficients of the cointegrating relationship $BR_t = \beta_1 + \beta_2 FFR_t + e_t$ at the same time as we estimated the other coefficients. Return to that example and estimate the error correction model with the cointegrating relationship replaced by the lagged residuals $\hat{e}_{t-1} = BR_{t-1} - 1.328 - 0.832FFR_{t-1}$. Compare your estimates with those obtained in Example 12.9, reported in equation (12.32).

12.20 The data file *canada6* contains monthly Canadian/U.S. exchange rates for the period 1971M1 to 2017M3. Split the observations into two sample periods—a 1971M1–1987M12 sample period and a 1988M1–2017M3 sample period.

a. Perform a unit root test on the data for each sample period. Which Dickey–Fuller tests did you use?

b. Are the results for the two sample periods consistent?

c. Perform a unit root test for the full sample 1971M1–2017M3. What is the order of integration of the data?

12.21 The data file *csi* contains the Consumer Sentiment Index (CSI) produced by the University of Michigan for the sample period 1978M1–2006M12.

a. Perform all three Dickey–Fuller tests. Are the results consistent? If not, why not?

b. Based on a graphical inspection of the data, which test should you have used?

c. Does the CSI suggest that consumers "remember" and "retain" news information for a short time, or for a long time?

12.22 The data file *mexico* contains real GDP for Mexico and the United States from the first quarter of 1980 to the third quarter of 2006. Both series have been standardized so that the average value in 2,000 is 100.

a. Perform the test for cointegration between Mexico and the Unites States using all three test equations in (12.29). Are the results consistent?

b. The theory of convergence in economic growth suggests that the two GDPs should be proportional and cointegrated. That is, there should be a cointegrating relationship that does not contain an intercept or a trend. Do your results support this theory?

c. If the variables are not cointegrated, what should you do if you are interested in testing the relationship between Mexico and the United States?

12.23 The data file *inter2* contains 300 observations of a generated I(2) process shown in Figure 12.10. Show that the variable called *inter2* is indeed an I(2) variable by conducting a number of unit root tests—first on the level of the data, then on the first difference, and finally on the second difference.

12.24 Prices around the world tend to move together. The data file *ukpi* contains information about the price indices in the United Kingdom and in the Euro Area (the United Kingdom is a member of the European Union, but not a member of the single European currency zone) for the period 1996M1–2009M12.

a. Plot the data. Are the series I(1) or I(0)?

b. Are prices in the UK and in the Euro Area cointegrated, or spuriously related? Use both the least squares and the error correction method to test this proposition.

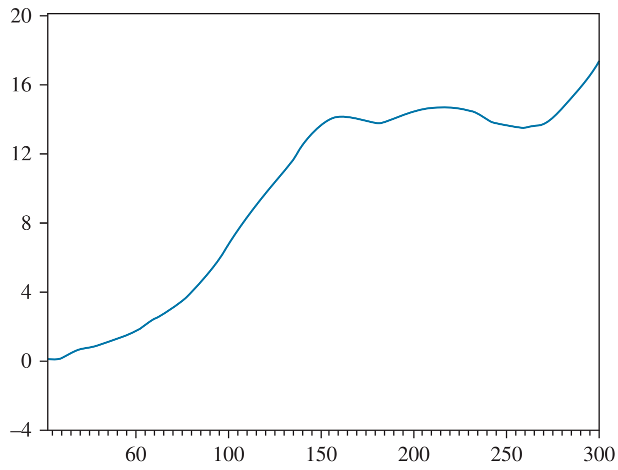

FIGURE 12.10 **A generated I(2) process.**

12.25 The data file *nasa* contains annual data on sunspots and the rate of growth in real GDP in the U.S. for the period 1950–2014. Jevons, a 19th century economist, suggested that there might be a relationship between business cycles and sunspots because variations in sunspots indicate variations in weather which in turn causes variation in agricultural output.

a. Plot each of the series. Do business cycles tend to follow sunspot activity?
b. Using a 5% significance level, test whether each series is stationary.
c. Set up an ARDL model to test the hypothesis that sunspots can be used to predict business cycles in the U.S. Do your results support Jevons' theory?

12.26 The data file *shiller* contains the stock market data in the book "Irrational Exuberance" by Robert Shiller.[10] They comprise the monthly price and dividends of the S&P Index (in logs) for the sample period 1871M1–2015M9. Finance theory suggests a long-run relationship between dividends and the stock price.

a. Plot each of the series. Do they appear to be moving together?
b. Carry out an empirical analysis to investigate whether there is evidence of a long-run relationship between the two series. Use a 1% level of significance for all hypothesis tests.

12.27 How easy is it to forecast the Australian/U.S. dollar exchange rate? The data file *iron* contains monthly data on the iron ore price and the exchange rate from 2010M1 to 2016M12. In the questions that follow, use a 5% significance level for all hypothesis tests.

a. Plot the two series. Do they appear to move together?
b. Is the exchange rate stationary or nonstationary? What model best reflects the relationship between current and past exchange rates?
c. Is the iron ore price stationary or nonstationary?
d. Financial commentators have suggested that, given Australia's dependence on iron ore exports, its exchange rate follows movements in the iron ore price. Is there evidence to suggest these financial commentators are correct?
e. Can the iron ore price be used to help forecast the exchange rate?

12.28 The data file *inflation* contains quarterly observations on the inflation rates in Germany and France from 1990Q1 to 2014Q4. For any hypothesis tests in the following questions, use a 5% significance level.

a. Plot each of the series and comment on the plots.
b. Use unit root tests, checks for serial correlation in the errors and significance of coefficients to specify and estimate an equation relating Germany's current inflation rate to its past rates.

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[10] Robert Shiller, *Irrational Exuberence*, 3rd ed, 2016, Princeton University Press.

c. Use unit root tests, checks for serial correlation in the errors and significance of coefficients to specify and estimate an equation relating France's current inflation rate to its past rates.

d. Are the inflation rates in France and Germany cointegrated?

e. Specify and estimate an equation relating Germany's current exchange rate to past exchange rates in France and Germany.

12.29 Reconsider Example 6.20 where a logistic growth curve for the share of U.S. steel produced by electric arc furnace (EAF) technology was estimated. The data are stored in the data file *steel*. The curve is given by the equation

$$y_t = \frac{\alpha}{1 + \exp(-\beta - \delta t)} + e_t$$

a. Plot the series $y_t = EAF_t$. Does it give the appearance of being stationary or nonstationary? Does the logistic growth curve appear to be a good model for modeling its trend?

b. Using a 5% significance level, test the series $y_t = EAF_t$ for a unit root.

c. Estimate the equation by nonlinear least squares and plot the residuals. Do the residuals appear to be stationary. Test the residuals for a unit root.

d. Using a 5% significance level, test the series $\Delta y_t = \Delta EAF_t$ for a unit root.

e. Estimate a first-differenced version of the model and plot the residuals. Do the residuals appear to be stationary. Test the residuals for a unit root.

f. Based on your answers to the previous parts of this question, do you think $y_t = EAF_t$ is trend stationary? Compare the estimates from parts (c) and (e). Do you think the nonlinear least-squares estimates in part (c) are reliable?